**Uncertainties: Summary by Examples**

1. **The uncertainty assigned to a measurement** reflects the precision of the instrument used to obtain it. The minimum uncertainty can be quoted as half the smallest division of the instrument only when one can see clearly that a measurement could occur between two adjacent, smallest divisions.

2. **Measurements are quoted in scientific notation** in the following form: $4.3 \pm 0.2 \, m$. The uncertainty is given to one significant figure. The value of the measurement is rounded off so as to reflect the precision of the uncertainty.

3. **Calculations based on measurements**: If $t = 1.5 \times 10^4 \pm 0.4 \times 10^3$ (or $15 \pm 4$) and $y = 4.6 \pm 0.3$ are the experimental measurements, to calculate the value given by $p = ty^2$ then first calculate the main value:

   \[
   p_{\text{main}} = (15)(4.6)^2 = 317.4000
   \]

   **Using relative uncertainties:**

   - Calculate the relative uncertainties:
     \[
     \Delta t_{rel} = \frac{\Delta t}{t} \times 100\% = \frac{4}{15} \times 100\% = 26.6666667\%
     \]
     \[
     \Delta y_{rel} = \frac{\Delta y}{y} \times 100\% = \frac{0.3}{4.6} \times 100\% = 6.52173913\%
     \]
   - Calculate the relative uncertainty of $p$: \[
   \Delta p_{rel} = \sqrt{\left(\Delta t_{rel}\right)^2 + \left(2\Delta y_{rel}\right)^2}
   \]
     \[
     = \sqrt{(26.7\%)^2 + 4(6.52\%)^2} = 29.68574\%
     \]
   - Calculate the absolute uncertainty of $p$: \[
   \Delta p = \frac{p_{\text{main}}\Delta p_{rel}}{100\%} = \frac{317.4000(29.68547\%)}{100\%} = 94.223
   \]
   - Round off uncertainty to one significant figure: \[\Delta p = 94.223 \Rightarrow 90\]
   - Round off the main result to the same precision: \[p = 3.2 \times 10^2 \pm 9 \times 10^1\] or \[3.2 \times 10^2 \pm 29.7\%\]

   If the equation had been $p = t + y^2$ then absolute uncertainties must be used.

   - Calculate the main value of $p$: \[p = 15 + (4.6)^2 = 36.16\]
   - Calculate the absolute uncertainty of $p$ and round off uncertainty to one significant figure:
     \[
     \Delta p = \sqrt{(\Delta t)^2 + (2\Delta y)^2} = \sqrt{(4)^2 + 4(0.3)^2} = 4.0447 \Rightarrow 4
     \]
   - Round off the main result to the same precision: \[p = 36 \pm 4\] or \[\pm 11.1\%\]

   Note that the factor 2 is included in the uncertainty calculations because $y$ is squared in the equations for $p$.

   **Using the injection method:**

   - Maximize each measured value and recalculate: \[p_{\text{max},t} = (15+4)(4.6)^2 = 402.0400\] \[p_{\text{max},y} = (15+4.6+0.3)^2 = 360.1500\]
   - The uncertainty is then: \[
   \Delta p = \sqrt{(317.4000 - 402.0400)^2 + (317.4000 - 360.1500)^2} = 94.82347863
   \]
   - Round off the uncertainty to one significant figure: \[\Delta p = 94.82347863 \Rightarrow 90\]
   - Round off the main result to the same precision: \[p = 3.2 \times 10^2 \pm 9 \times 10^1\]

4. **The uncertainty of an average value**. The uncertainty of the average value of the following measurements; \[x = 12 \pm 2, \, x = 14 \pm 5, \, x = 13 \pm 3 \, \text{and} \, x = 14 \pm 2 \, \text{where} \, \frac{x_{\text{average}}}{4} = \frac{12 + 14 + 13 + 14}{4} = 13.25000000\].

   can be calculated using one of two methods:

   a) **The deviation from the mean**. The deviation of each of the measured values from the average value is calculated. These are then squared and summed. The uncertainty of the average is then calculated as the square root of this sum divided by $\sqrt{n(n-1)}$ where $n$ is the number of data points to give
So that the average value is quoted as: \( x_{\text{average}} = 13.2 \pm 0.5 \). The uncertainty of the average calculated this way is called the standard deviation of the mean.

b) **The injection method**: The square root of the sum of the squares of the individual uncertainties is computed and divided by the number of measurements. This gives

\[
\Delta x = \frac{\sqrt{(13.25 - 12)^2 + (13.25 - 14)^2 + (13.25 - 13)^2 + (13.25 - 14)^2}}{\sqrt{4 \times 3}} = \frac{\sqrt{2.75000000}}{\sqrt{12}} = 0.4787135539 \Rightarrow 0.5
\]

where 4 is the number of data points, so that the average is quoted as: \( x_{\text{average}} = 13 \pm 2 \) and where \( \Delta x \) has been rounded off to one significant figure 1.620185175 \( \Rightarrow 2 \).

Usually, the method that gives the largest uncertainty is used but in most cases the standard deviation from the mean is the most rigorous method.

5. **Fitting a straight line to a set of data points.**

Use a linear regression algorithm from Excel or any other software package. The full regression function will give the best slope along with its uncertainty as well as the best intercept along with its uncertainty. These uncertainties must be rounded off to 1 significant figure and the corresponding main values must be rounded off to the same precision as the uncertainties.

For a full discussion on uncertainties and their uses, go to

http://web2.slc.qc.ca/smanoli/docs/Measurements, Calculations and Uncertainties (Fall 2013).pdf