The Trace and Transpose of a Matrix

1. Let \( A = [a_{ij}]_{n \times n} \). We denote and define the **trace of \( A \)** by \( tr(A) = \sum_{k=1}^{n} (A)_{kk} = \sum_{k} (A)_{kk} \). In other words, the trace of \( A \) is the sum of its entries along the main diagonal.

   (a) Compute the trace of \( A \) in each case.

   i. \( A = \begin{bmatrix} 1 & 3 \\ 4 & -6 \end{bmatrix} \)
   
   ii. \( A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 2 \\ 0 & 1 & 4 \end{bmatrix} \)
   
   iii. \( A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 3 & 4 & 0 \\ 7 & 4 & -5 & 8 \\ 6 & 0 & 2 & 1 \end{bmatrix} \)

   (b) Show that \( tr(cA) = c \cdot tr(A) \), for any \( c \in \mathbb{R} \).

   (c) Show that \( tr(A + B) = tr(A) + tr(B) \), for any \( A, B \in \mathbb{R}^{n \times n} \).

   (d) Show that \( tr(AB) = tr(BA) \), for any \( A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times m} \). Why is this not obvious?

   (e) Show that \( tr(AB - BA) = 0 \), for any \( A, B \in \mathbb{R}^{n \times n} \).

   (f) Describe explicitly all \( 2 \times 2 \) matrices with zero trace. Write your generic matrix as a linear combination of three matrices. (There are two possible natural answers in this case.)

2. Let \( A \in \mathbb{R}^{m \times n} \). The **transpose of \( A \)** is denoted by \( A^T \in \mathbb{R}^{n \times m} \) and is defined by the equality \((A^T)_{ij} = (A)_{ji}\). In other words, the transpose of \( A \) is obtained by interchanging the rows and columns of \( A \). Compute the transpose of \( A \) in each case, that is, find \( A^T \).

   (a) \( A = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix} \)
   
   (b) \( A = \begin{bmatrix} 8 & 4 & 3 & -1 \\ 0 & -2 & 5 & 9 \end{bmatrix} \)
   
   (c) \( A = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 6 \end{bmatrix} \)
   
   (d) \( A = \begin{bmatrix} 6 & -3 & 2 \\ 1 & 5 & 3 \\ 4 & 9 & 7 \end{bmatrix} \)
   
   (e) \( A = \begin{bmatrix} 5 & -4 & -3 \end{bmatrix} \)
   
   (f) \( A = \begin{bmatrix} 1 & 2 & 3 & -5 \\ 0 & 1 & -8 & 9 \\ -6 & 6 & 5 & 0 \end{bmatrix} \)
3. Prove the following properties of the transpose. Assume that addition/multiplication is defined where applicable.

(a) \((A^T)^T = A\).
(b) \((kA)^T = kA^T\), for any \(k \in \mathbb{R}\).
(c) \((A \pm B)^T = A^T \pm B^T\).
(d) \((AB)^T = B^T A^T\).
(e) \((A_1 A_2 \cdots A_m)^T = A_m^T \cdots A_2^T A_1^T\). (Apply (d) recursively.)
(f) \((A^n)^T = (A^T)^n\), for any \(n \in \mathbb{N}\). (Apply (e) directly.)

4. A square matrix \(A\) is said to be **symmetric** if \(A^T = A\) and **skew-symmetric** if \(A^T = -A\).

(a) Given a square matrix \(A\), show that the following matrices are symmetric: \(AA^T, A^T A\) and \(A + A^T\).
(b) Describe all \(2 \times 2\) symmetric matrices explicitly and express your generic symmetric matrix as a linear combination of three matrices.
(c) Given a square matrix \(A\), show that \(A - A^T\) is skew-symmetric.
(d) Describe all \(3 \times 3\) skew-symmetric matrices explicitly and express your generic skew-symmetric matrix as a linear combination of three matrices.
(e) Show that any square matrix \(A\) can be expressed as the sum of a symmetric and a skew-symmetric matrix. (Hint: Consider \(A + A^T\) and \(A - A^T\).)

---

### Answers

1. (a) i. \(tr(A) = 1 - 6 = -5\)  
   ii. \(tr(A) = 1 + 3 + 4 = 8\)  
   iii. \(tr(A) = 1 + 3 - 5 + 1 = 0\)

(b) \(tr(cA) = \sum_k (cA)_{kk} = \sum_k c(A)_{kk} = c \sum_k (A)_{kk} = c \cdot tr(A)\).

(c) \(tr(A \pm B) = \sum_k (A \pm B)_{kk} = \sum_k ((A)_{kk} \pm (B)_{kk}) = \sum_k (A)_{kk} \pm \sum_k (B)_{kk} = tr(A) \pm tr(B)\).

(d) \(tr(AB) = \sum_k (AB)_{kk} = \sum_k \sum_\ell (A)_{k\ell} (B)_{\ell k} = \sum_\ell \sum_k (A)_{k\ell} (B)_{\ell k} = \sum_\ell \sum_k (B)_{\ell k} (A)_{k\ell} = \sum_\ell (BA)_{\ell \ell} = tr(BA)\).

   This is not obvious since \(AB\) and \(BA\) may not be of the same size. Furthermore, even if \(AB\) and \(BA\) are of the same size, we know that \(AB \neq BA\) in general.

(e) Using properties (c) and (d), we have that \(tr(AB - BA) = tr(AB) - tr(BA) = 0\).

(f) \[
\begin{bmatrix}
    a & b \\
    c & -a
\end{bmatrix} = a \begin{bmatrix}
    1 & 0 \\
    0 & -1
\end{bmatrix} + b \begin{bmatrix}
    0 & 1 \\
    0 & 0
\end{bmatrix} + c \begin{bmatrix}
    0 & 0 \\
    1 & 0
\end{bmatrix}
\]
2. (a) \( A^T = \begin{bmatrix} 1 & -4 \\ 2 & 5 \end{bmatrix} \)  
(b) \( A^T = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 5 \\ -1 & 9 \end{bmatrix} \)  
(c) \( A^T = \begin{bmatrix} 1 & 0 & 1 & 6 \end{bmatrix} \)  
(d) \( A^T = \begin{bmatrix} 6 & 1 & 4 \\ -3 & 5 & 9 \\ 2 & 3 & 7 \end{bmatrix} \)  
(e) \( A^T = \begin{bmatrix} 5 \\ -4 \\ -3 \end{bmatrix} \)  
(f) \( A^T = \begin{bmatrix} 1 & 0 & -6 \\ 2 & 1 & 6 \\ 3 & -8 & 5 \\ -5 & 9 & 0 \end{bmatrix} \)

3. (a) \( ((A^T)^T)_{ij} = (A^T)_{ji} = (A)_{ij} \)  
(b) \( ((kA)^T)_{ij} = (kA)_{ji} = k(A)_{ji} = k(A^T)_{ij} = (kA^T)_{ij} \)  
(c) \( ((A \pm B)^T)_{ij} = (A \pm B)_{ji} = (A)_{ji} \pm (B)_{ji} = (A^T)_{ij} \pm (B^T)_{ij} = (A^T \pm B^T)_{ij} \)  
(d) \( ((AB)^T)_{ij} = (AB)_{ji} = \sum_k (A)_{jk}(B)_{ki} = \sum_k (A^T)_{kj}(B^T)_{ik} = \sum_k (B^T)_{ik}(A^T)_{kj} = (B^T A^T)_{ij} \)  
(e) Apply (d) recursively.  
(f) Write \( A^n = \overbrace{AA \cdots A}^{n \text{ times}} \) and apply (e) directly.

4. (a) Show this directly using properties of the transpose.  
(b) All 2 \times 2 symmetric matrices are of the form  
\[
\begin{bmatrix} a & b \\ b & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]  
(c) Show this directly using properties of the transpose.  
(d) All 3 \times 3 skew-symmetric matrices are of the form  
\[
\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = a \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}
\]  
(e) It’s almost enough to add up the two matrices provided in the hint.