The Trace and Transpose of a Matrix

1. Let \( A = [a_{ij}]_{n \times n} \). We denote and define the trace of \( A \) by \( \text{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} \).
   In other words, the trace of \( A \) is the sum of its entries along the main diagonal.

   (a) Compute the trace of \( A \) if
   
   \( i. \) \( A = \begin{bmatrix} 1 & 3 \\ 4 & -6 \end{bmatrix} \)  
   \( ii. \) \( A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 2 \\ 0 & 1 & 4 \end{bmatrix} \)  
   \( iii. \) \( A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 3 & 4 & 0 \\ 7 & 4 & -5 & 8 \\ 6 & 0 & 2 & 1 \end{bmatrix} \)

   (b) Show that \( \text{tr}(kA) = k \cdot \text{tr}(A) \), for any \( k \in \mathbb{R} \).

   (c) Show that \( \text{tr}(A \pm B) = \text{tr}(A) \pm \text{tr}(B) \), for any \( A, B \in \mathbb{R}^{n \times n} \).

   (d) (Optional) Show that \( \text{tr}(AB) = \text{tr}(BA) \), for any \( A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times m} \).
   Why is this not obvious?

   (e) Show that it is impossible to find matrices \( A, B \in \mathbb{R}^{n \times n} \) such that \( AB - BA = I_n \), where \( I_n \) is the \( n \times n \) identity matrix. (Hint: Use the trace.)

   (f) Describe explicitly all \( 2 \times 2 \) matrices with zero trace. Write your generic matrix as a linear combination of three matrices. (There are two possible natural answers in this case.)

2. Let \( A \in \mathbb{R}^{m \times n} \). The transpose of \( A \) is denoted by \( A^T \in \mathbb{R}^{n \times m} \) and is defined by the equality \( (A^T)_{ij} = (A)_{ji} \). In other words, the transpose of \( A \) is obtained by interchanging the rows and columns of \( A \). Compute the transpose of \( A \) in each case, that is, find \( A^T \).

   (a) \( A = \begin{bmatrix} 1 & 2 \\ -4 & 5 \end{bmatrix} \)  
   (d) \( A = \begin{bmatrix} 6 & -3 & 2 \\ 1 & 5 & 3 \\ 4 & 9 & 7 \end{bmatrix} \)

   (b) \( A = \begin{bmatrix} 8 & 4 & 3 & -1 \\ 0 & -2 & 5 & 9 \end{bmatrix} \)  
   (e) \( A = \begin{bmatrix} 5 & -4 & -3 \end{bmatrix} \)

   (c) \( A = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 6 \end{bmatrix} \)  
   (f) \( A = \begin{bmatrix} 1 & 2 & 3 & -5 \\ 0 & 1 & -8 & 9 \\ -6 & 6 & 5 & 0 \end{bmatrix} \)

3. State whether the following properties of the transpose are true or false. (Assume that addition/multiplication is defined where applicable.)

   (a) \( (A^T)^T = A \).

   (b) \( (kA)^T = kA^T \), for any \( k \in \mathbb{R} \).

   (c) \( (A \pm B)^T = A^T \pm B^T \).

   (d) \( (AB)^T = B^T A^T \).

   (e) \( (A_1 A_2 \cdots A_m)^T = A_m^T \cdots A_2^T A_1^T \).

   (f) \( (A^n)^T = (A^T)^n \), for any \( n \in \mathbb{N} \).
4. A square matrix \( A \) is said to be symmetric if \( A^T = A \) and skew-symmetric if \( A^T = -A \).

(a) Given a square matrix \( A \), show that the following matrices are symmetric: \( AA^T \), \( A^TA \) and \( A + A^T \).

(b) Describe all \( 2 \times 2 \) symmetric matrices explicitly and express your generic symmetric matrix as a linear combination of three matrices.

(c) Given a square matrix \( A \), show that \( A - A^T \) is skew-symmetric.

(d) Describe all \( 3 \times 3 \) skew-symmetric matrices explicitly and express your generic skew-symmetric matrix as a linear combination of three matrices.

(e) Show that any square matrix \( A \) can be expressed as the sum of a symmetric and a skew-symmetric matrix. (Hint: Consider \( A + A^T \) and \( A - A^T \).)

Answers

1. (a) i. \( \text{tr}(A) = 1 - 6 = -5 \) ii. \( \text{tr}(A) = 1 + 3 + 4 = 8 \) iii. \( \text{tr}(A) = 1 + 3 - 5 + 1 = 0 \)

(b) \( \text{tr}(kA) = ka_{11} + ka_{22} + \cdots + ka_{nn} = k(a_{11} + a_{22} + \cdots + a_{nn}) = k \cdot \text{tr}(A) \)

(c) \( \text{tr}(A \pm B) = (a_{11} \pm b_{11}) + (a_{22} \pm b_{22}) + \cdots + (a_{nn} \pm b_{nn}) = (a_{11} + a_{22} + \cdots + a_{nn}) \pm (b_{11} + b_{22} + \cdots + b_{nn}) = \text{tr}(A) \pm \text{tr}(B) \)

(d) Use sigma notation and sum the terms in a different order.

This is not obvious since \( AB \) and \( BA \) may not be of the same size. Furthermore, even if \( AB \) and \( BA \) are of the same size, we know that \( AB \neq BA \) in general.

(e) The trace on the left is \( \text{tr}(AB - BA) = \text{tr}(AB) - \text{tr}(BA) = 0 \) and the trace on the right is \( \text{tr}(I_n) = n \geq 1 \). If both \( AB - BA \) and \( I_n \) were to be equal, then they would have to have the same trace, but this is not possible as we have just shown.

(f) \( \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \)

2. (a) \( A^T = \begin{bmatrix} 1 & -4 \\ 2 & 5 \end{bmatrix} \)

(b) \( A^T = \begin{bmatrix} 8 & 4 & 1 \\ 0 & -2 & 3 \\ -1 & 5 & 9 \end{bmatrix} \)

(c) \( A^T = \begin{bmatrix} 1 & 0 & 1 & 0 & 6 \end{bmatrix} \)

(d) \( A^T = \begin{bmatrix} 6 & 1 & 4 \\ -3 & 5 & 9 \\ 2 & 3 & 7 \end{bmatrix} \)

(e) \( A^T = \begin{bmatrix} 5 \\ -4 \\ -3 \end{bmatrix} \)

(f) \( A^T = \begin{bmatrix} 1 & 0 & -6 \\ 2 & 1 & 6 \\ 3 & -8 & 5 \\ -5 & 9 & 0 \end{bmatrix} \)

3. All properties are true!
4. (a) \((A A^T)^T = (A^T)^T A^T = AA^T\), so \(AA^T\) is symmetric.
\((A^T A)^T = A^T (A^T)^T = A^T A\), so \(A^T A\) is symmetric.
\((A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T\), so \(A + A^T\) is symmetric.

(b) All 2 \(\times\) 2 symmetric matrices are of the form
\[
\begin{bmatrix}
a & b \\
b & d
\end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

(c) \((A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T)\), so \(A - A^T\) is skew-symmetric.

(d) All 3 \(\times\) 3 skew-symmetric matrices are of the form
\[
\begin{bmatrix}
0 & a & b \\
-a & 0 & c \\
-b & -c & 0
\end{bmatrix} = a \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}
\]

(e) \(A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)\)

This does the trick as the first matrix is symmetric and the second matrix is skew-symmetric.