Additional Problems

1. Find all $2 \times 2$ matrices that commute with $\begin{bmatrix} 1 & 2 \\ -4 & 3 \end{bmatrix}$.

2. A jar containing quarters, loonies, and toonies has twenty coins with a total value of twenty-four dollars. Find all possible combinations or coins. (Be systematic!)

3. Show that if $A \in \mathbb{R}^{n \times n}$ is invertible, then the linear system $A^k X = 0$ has only the trivial solution for any $k \in \mathbb{Z}$.

4. Let $V \in \mathbb{R}^{n \times 1}$ be such that $V^T V = 1$. Let $H = I - 2VV^T$. Show that $H$ is symmetric and that it is its own inverse.

5. Solve the given linear systems simultaneously, that is, by reducing a single augmented matrix.

\[
\begin{align*}
\text{(a)} & \quad x + 6y - 3z = 1 \\
& \quad 3x + y = 2 \\
& \quad -x - 8y + 4z = 5 \\
\end{align*}
\]

\[
\begin{align*}
\text{(b)} & \quad x + 6y - 3z = -4 \\
& \quad 3x + y = -1 \\
& \quad -x - 8y + 4z = 1 \\
\end{align*}
\]

6. Solve the given linear systems simultaneously, that is, by reducing a single augmented matrix.

\[
\begin{align*}
\text{(a)} & \quad x - 2y + 3z = -1 \\
& \quad 4x + y - z = -1 \\
& \quad -7x - 4y + 5z = 1 \\
\end{align*}
\]

\[
\begin{align*}
\text{(b)} & \quad x - 2y + 3z = 1 \\
& \quad 4x + y - z = 1 \\
& \quad -7x - 4y + 5z = 1 \\
\end{align*}
\]

\[
\begin{align*}
\text{(c)} & \quad x - 2y + 3z = -1 \\
& \quad 4x + y - z = 1 \\
& \quad -7x - 4y + 5z = -3 \\
\end{align*}
\]

7. Find conditions on $a, b, c \in \mathbb{R}$ such that the given linear system is consistent.

\[
\begin{align*}
x - 2y + 3z & = a \\
4x + y - z & = b \\
-7x - 4y + 5z & = c \\
\end{align*}
\]

8. State whether the following statements are true or false. Be sure to justify.

(a) If $A$ and $B$ are symmetric, then so is $AB$.

(b) If $A$ is invertible and symmetric, then $A^{-1}$ is also symmetric.

(c) $AA^T$ and $A^T A$ are symmetric for any $A \in \mathbb{R}^{m \times n}$.

(d) If $A \in \mathbb{R}^{n \times n}$ is symmetric, then $p(A)$ is also symmetric for any polynomial $p(x)$.

9. Let $A \in \mathbb{R}^{n \times n}$. Show that there exists an invertible matrix $B$ and an upper triangular matrix $U$ such that $A = BU$.

10. Given $A$, find an invertible matrix $B$ and an upper triangular matrix $U$ such that $A = BU$.

\[
\begin{align*}
\text{(a)} & \quad A = \begin{bmatrix} 2 & 10 \\ 1 & 7 \end{bmatrix} \\
\text{(b)} & \quad A = \begin{bmatrix} -1 & 1 & 2 \\ 2 & -1 & -2 \\ 3 & 2 & 4 \end{bmatrix}
\end{align*}
\]
11. (Optional)
Recall that if $A, B \in \mathbb{R}^{n \times n}$ are invertible, then $AB$ is also invertible since $(AB)^{-1} = B^{-1}A^{-1}$.
In this problem, you will show that the converse is also true, that is:

Show that if $AB$ is invertible, then both $A$ and $B$ must also be invertible.

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**Answers**

1. We are looking for matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfying $\begin{bmatrix} 1 & 2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -4 & 3 \end{bmatrix}$.

   Multiplying both sides and equating corresponding entries yields a linear system.

   Solve the linear system to obtain $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} t & s \\ -2s & t + s \end{bmatrix}, \quad t, s \in \mathbb{R}$.

2. $(4, 9, 7)$ or $(8, 2, 10)$

3. Since $A$ is invertible, then so is $A^k$. $A^kX = 0 \iff A^{-k}A^kX = A^{-k}0 \iff X = 0$ is the only solution.

4. It is routine to verify that $H^T = H$ and $HH = I$.

5. (a) $x = 19, y = -55, z = -104$
   (b) $x = -13, y = 38, z = 73$

6. (a) $\begin{cases} x = \frac{1}{3} - t \\ y = \frac{1}{3} + 13t \\ z = 9t \end{cases}$
   ; $t \in \mathbb{R}$

   (c) $\begin{cases} x = \frac{1}{9} - t \\ y = \frac{5}{9} + 13t \\ z = 9t \end{cases}$
   ; $t \in \mathbb{R}$

7. $a = 2b + c$

8. (a) False
   (b) True
   (c) True
   (d) True

9. Use row reduction and elementary matrices.

10. (a) $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$
    or $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 0 & -4 \end{bmatrix}$

    (b) $A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ 3 & 5 & 1 \end{bmatrix}$

11. First, show that $B$ is invertible using the fact that $B$ is invertible if and only if the linear system $BX = 0$ has only the trivial solution. Then, combining this with the fact that $AB$ is invertible by assumption, you should have no problem showing that $A$ is also invertible.