**Additional Problems**

1. Find all $2 \times 2$ matrices that commute with $\begin{bmatrix} 1 & 2 \\ -4 & 3 \end{bmatrix}$.

2. A jar containing quarters, loonies, and toonies has twenty coins with a total value of twenty-four dollars. Find all possible combinations or coins. (Be systematic!)

3. Show that if $A \in \mathbb{R}^{n \times n}$ is invertible, then the linear system $A^k X = 0$ has only the trivial solution for any $k \in \mathbb{Z}$.

4. Let $V \in \mathbb{R}^{n \times 1}$ be such that $V^T V = 1$. Let $H = I - 2VV^T$. Show that $H$ is symmetric and that it is its own inverse.

5. Solve the given linear systems simultaneously, that is, by reducing a single augmented matrix.

   $\begin{align*}
   x + 6y - 3z &= 1 \\
   3x + y &= 2 \\
   -x - 8y + 4z &= 5 \\
   \end{align*}$

   $\begin{align*}
   x + 6y - 3z &= -4 \\
   3x + y &= -1 \\
   -x - 8y + 4z &= 1 \\
   \end{align*}$

6. Solve the given linear systems simultaneously, that is, by reducing a single augmented matrix.

   $\begin{align*}
   x - 2y + 3z &= -1 \\
   4x + y - z &= -1 \\
   -7x - 4y + 5z &= 1 \\
   \end{align*}$

   $\begin{align*}
   x - 2y + 3z &= 1 \\
   4x + y - z &= 1 \\
   -7x - 4y + 5z &= 1 \\
   \end{align*}$

   $\begin{align*}
   x - 2y + 3z &= -1 \\
   4x + y - z &= 1 \\
   -7x - 4y + 5z &= -3 \\
   \end{align*}$

7. Find conditions on $a, b, c \in \mathbb{R}$ such that the given linear system is consistent.

   $\begin{align*}
   x - 2y + 3z &= a \\
   4x + y - z &= b \\
   -7x - 4y + 5z &= c \\
   \end{align*}$

8. State whether the following statements are true or false. Be sure to justify.

   (a) If $A$ and $B$ are symmetric, then so is $AB$.

   (b) If $A$ is invertible and symmetric, then $A^{-1}$ is also symmetric.

   (c) $AA^T$ and $A^TA$ are symmetric for any $A \in \mathbb{R}^{m \times n}$.

   (d) If $A \in \mathbb{R}^{n \times n}$ is symmetric, then $p(A)$ is also symmetric for any polynomial $p(x)$.

9. Let $A \in \mathbb{R}^{n \times n}$. Show that there exists an invertible matrix $B$ and an upper triangular matrix $U$ such that $A = BU$.

10. Given $A$, find an invertible matrix $B$ and an upper triangular matrix $U$ such that $A = BU$.

   (a) $A = \begin{bmatrix} 2 & 10 \\ 1 & 7 \end{bmatrix}$

   (b) $A = \begin{bmatrix} -1 & 1 & 2 \\ 2 & -1 & -2 \\ 3 & 2 & 4 \end{bmatrix}$
Answers

1. We are looking for matrices \[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\] satisfying \[
\begin{bmatrix}
  1 & 2 \\
  -4 & 3
\end{bmatrix}
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
= \begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\begin{bmatrix}
  1 & 2 \\
  -4 & 3
\end{bmatrix}.
\]

Multiplying both sides and equating corresponding entries yields a linear system.

Solve the linear system to obtain \[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
= \begin{bmatrix}
  t + s & -s \\
  2s & t
\end{bmatrix}, \quad t, s \in \mathbb{R}.
\]

2. First, solve the linear system \[
x + y + z = 20 \\
\frac{1}{4}x + y + 2z = 24
\]

Then, restrict the values of your parameter to guarantee that \(x, y, z\) are non-negative integers.

This will lead to two possible solutions: \((4, 9, 7)\) or \((8, 2, 10)\)

3. Since \(A\) is invertible, then so is \(A^k\).

\[A^kX = 0 \implies A^{-k}A^kX = A^{-k}0 \implies X = 0\] is the only solution.

4. It is straightforward to verify that \(H^T = H\) and \(HH = I\).

5. (a) \(x = 19, y = -55, z = -104\)  
(b) \(x = -13, y = 38, z = 73\)

\[
\begin{cases}
  x = \frac{1}{3} - t \\
  y = \frac{1}{3} + 13t \\
  z = 9t
\end{cases} \quad \text{; } t \in \mathbb{R}
\]

6. (a) \(x = \frac{1}{3} - t\)  
(b) \(\emptyset\)  
(c) \[
\begin{cases}
  x = \frac{1}{9} - t \\
  y = \frac{5}{9} + 13t \\
  z = 9t
\end{cases} \quad \text{; } t \in \mathbb{R}
\]

7. Row-reducing the linear system should lead you to the following conclusion: \(a = 2b + c\)

8. (a) False: \((AB)^T = B^T A^T = BA \neq AB\) in general.  
(b) True: \((A^{-1})^T = (A^T)^{-1} = A^{-1}\).  
(c) True: see problem sheet #4, problem 4-(a).  
(d) True: Let \(p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m\) and show that \((p(A))^T = p(A)\).

9. Row reduce the augmented matrix \([A|I]\) to \([U|C]\), where \(U\) is an upper triangular matrix. We know that \(CA = U\), so \(A = C^{-1}U\). Letting \(B = C^{-1}\) yields \(A = BU\), which completes the proof.

10. (a) \[A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix} \text{ or } \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 0 & -4 \end{bmatrix}\]
    (b) \[A = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ 3 & 5 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}\]