Gaussian & Gauss-Jordan Elimination

1. In each case, find two different row-echelon forms of the given matrix.

\[
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
-1 & 1 \\
2 & 4 \\
\end{pmatrix}
\]

2. Would the previous problem be possible if the term “row-echelon” was replaced by the term “reduced row-echelon”? 

3. In each case, determine whether the matrix is in row-echelon form, reduced row-echelon form, or neither.

\[
\begin{pmatrix}
1 & 3 & 0 \\
0 & 1 & 2 \\
0 & 0 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 2 & 0 & 4 \\
0 & 0 & 1 & -3 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 & 2 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

4. Find the solution set of the linear system of equations. (Use the method of your choice.)

\[
\begin{align*}
\text{(a)} & \quad x + 2y &= -3 \\
& \quad 3x + 4y &= 1 \\
& \quad 2x + 4y - 6z &= -1 \\
\text{(b)} & \quad 5x + 3y + z &= 0 \\
& \quad 3x - y + 7z &= 1 \\
\end{align*}
\]

\[
\begin{align*}
\text{(c)} & \quad x - z &= 4 \\
& \quad 3x + y - z &= 15 \\
& \quad x + y + z &= 6 \\
\text{(d)} & \quad 2x_1 - x_3 + 2x_4 &= -2 \\
& \quad 3x_1 + x_2 - x_4 &= 1 \\
\end{align*}
\]

5. Is it possible that a homogeneous linear system of equations has no solutions?

6. Determine whether the homogeneous linear system has one or infinitely many solutions.
(Do as little work as possible.)

\[
\begin{align*}
\text{(a)} & \quad \pi x - y + 3z &= 0 \\
& \quad \sqrt{2}x + 2y + 4z &= 0 \\
\text{(c)} & \quad x - 2y + 3z &= 0 \\
& \quad 3x + y + 5z &= 0 \\
\text{(b)} & \quad x_1 + 3x_2 - 5x_3 + 6x_4 &= 0 \\
& \quad 2x_1 - x_2 + 2x_4 &= 0 \\
\text{(d)} & \quad x - y + 4z &= 0 \\
& \quad 4x + y + 9z &= 0 \\
\end{align*}
\]
7. In each case, suppose that the augmented matrix associated to a linear system has been reduced using elementary row operations to the given row-echelon form. Solve the system using backward substitution. (Always classify the free variables as parameters, and then solve for the leading variables.)

(a) \[
\begin{bmatrix}
1 & 4 & 3 \\
0 & 1 & -2
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
1 & -5 & 8 \\
0 & 0 & 0
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
1 & 2 & 7 \\
0 & 0 & -3
\end{bmatrix}
\]

(d) \[
\begin{bmatrix}
1 & 2 & 3 & 9 \\
0 & 1 & -4 & -5 \\
0 & 0 & 1 & -1
\end{bmatrix}
\]

(e) \[
\begin{bmatrix}
1 & -2 & -1 & 4 \\
0 & 1 & 5 & 6 \\
0 & 0 & 1 & -3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

8. In each case, suppose that the augmented matrix associated to a linear system has been reduced using elementary row operations to the given reduced row-echelon form. Solve the system. (You should be able to write down the solution set directly without having to perform any additional calculations.)

(a) \[
\begin{bmatrix}
1 & 0 & 0 & -5 \\
0 & 1 & 0 & 8 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
1 & -2 & 6 & 7 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
1 & 4 & 0 & -1/2 \\
0 & 0 & 1 & 1/3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

9. Use Gaussian elimination or Gauss-Jordan elimination to solve the following linear systems. (When row reducing, try to avoid introducing fractions as much as possible.)

(a) 
\[-x + 2y = -2 \]
\[5x - 11y = 1 \]

(b) 
\[4x + y = -2 \]
\[3x + 2y = 5 \]
\[-2x + y - z = 6 \]

(c) 
\[-3x - y - z = 1 \]
\[x - 2y + z = 2 \]
\[x + 2y + 3z = -4 \]

(d) 
\[2x + y + z = 6 \]
\[5x + 4y + 5z = 11 \]
\[-x + y + 3z = 4 \]
\[-7x + 6y + 2z = 1 \]

(e) 
\[6x - 5y + z = 3 \]

(f) 
\[2x_1 + x_2 - x_3 = 3 \]
\[-x_1 - x_2 + 3x_3 + 4x_4 = -7 \]
\[6x_1 + 2x_2 + x_3 + x_4 = -1 \]

(g) 
\[4x_1 + 7x_2 + 4x_3 = 4 \]
\[-x_1 - x_3 + 2x_4 = -3 \]

(h) 
\[2x_1 + 3x_2 + x_3 + x_4 = -1 \]
\[x_1 - 11x_3 + 2x_4 = 2 \]
\[-x_1 + 6x_2 + 2x_3 + x_4 = 1 \]
10. For which values of $k$ does the linear system have no solutions, a unique solution, or infinitely many solutions?

(a) \[
\begin{align*}
  kx - 2y &= 3k + 2 \\
  x + (k - 3)y &= 4
\end{align*}
\]
(b) \[
\begin{align*}
  -2x + (k - 3)y + kz &= k + 11 \\
  (k + 1)y - z &= 5 \\
  x + 2y + z &= -1
\end{align*}
\]
(c) \[
\begin{align*}
  x + 2y - 3z &= 4 \\
  3x - y + 5z &= 2 \\
  4x + y + (k^2 - 14)z &= k + 2
\end{align*}
\]
(d) \[
\begin{align*}
  (k^2 - 9)x &= k + 3 \\
  (k - 5)y &= 1 \\
  (k + 8)z &= -10
\end{align*}
\]

11. (a) Find the equation of the cubic polynomial passing through the points $(0, -1), (1, 0), (-1, -4)$ and $(2, 5)$.

(b) (Optional) Show that the equation of a circle can be expressed in the form $x^2 + y^2 + ax + by = c$, for some real numbers $a, b, c \in \mathbb{R}$.

(c) (Optional) Explain why three points on a circle are required to determine its equation. Find the equation of the circle passing through the points $(0, -1), (2, 1)$ and $(3, -1)$. Provide its center and radius.

12. Use a proper change of variables to solve the given nonlinear system of equations. How many solutions are there?

(a) \[
\begin{align*}
  -2\sin(\theta) + 5x^2 &= 45, \text{ where } 0 \leq \theta \leq 2\pi. \\
  -\sin(\theta) + 2x^2 &= 18
\end{align*}
\]
(b) \[
\begin{align*}
  -2e^x + \frac{1}{y^2} + \frac{2}{z} &= -12 \\
  4e^x - \frac{1}{y^2} - \frac{3}{z} &= 23
\end{align*}
\]

13. (Optional)

(a) Show that if $ad - bc \neq 0$, then the reduced row-echelon form of \[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\]
is \[
\begin{bmatrix}
  1 & 0 \\
  0 & 1
\end{bmatrix}.
\]

(b) Show that if $ad - bc \neq 0$, then the linear system \[
\begin{align*}
  ax + by &= e \\
  cx + dy &= f
\end{align*}
\]
has a unique solution for any $e, f \in \mathbb{R}$. 
Answers

1. (a) \[
\begin{bmatrix}
1 & 2 \\
0 & 1
\end{bmatrix}
\] and \[
\begin{bmatrix}
1 & 4/3 \\
0 & 1
\end{bmatrix}
\] (b) \[
\begin{bmatrix}
1 & -1 \\
0 & 1
\end{bmatrix}
\] and \[
\begin{bmatrix}
1 & 2 \\
0 & 1
\end{bmatrix}
\]

2. No. The reduced row-echelon form of a matrix is unique.

3. (a) r.e.f. (d) r.r.e.f.
(b) r.r.e.f. (e) neither
(c) neither (f) r.r.e.f.
(g) r.e.f. (h) neither
(i) r.e.f.

4. (a) \[
\begin{cases}
x = 7 \\
y = -5
\end{cases}
\]
(b) \[
\begin{cases}
x = \frac{3}{14} - 11t \\
y = \frac{-5}{14} + 16t \\
z = 7t
\end{cases}; \quad t \in \mathbb{R}
\]
\[
\begin{cases}
x_1 = -1 + r - s \\
x_2 = 4 - 3r + 4s \\
x_3 = 2r \\
x_4 = s
\end{cases}; \quad r, s \in \mathbb{R}
\]
(e) \[
\begin{cases}
x = -6/5 \\
y = 16/5 \\
z = -7/5
\end{cases}
\]

5. No. A homogeneous linear system always has at least the trivial solution \( x_1 = x_2 = \cdots = x_n = 0 \).

6. (a) The system has infinitely many solutions. No work is required as there are 3 variables but only 2 equations.
(b) The system has infinitely many solutions. No work is required as there are 4 variables but only 3 equations.
(c) The system has only the trivial solution. (Work is required.)
(d) The system has infinitely many solutions. (Work is required.)

7. (a) \[
\begin{cases}
x = 11 \\
y = -2
\end{cases}
\]
(b) \[
\begin{cases}
x = 8 + 5t \\
y = t
\end{cases}; \quad t \in \mathbb{R}
\]
(c) \[ \emptyset \]
(d) \[
\begin{cases}
x = 30 \\
y = -9 \\
z = -1
\end{cases}
\]
(e) \[
\begin{cases}
x = 19 - 5t \\
y = t \\
z = 2
\end{cases}; \quad t \in \mathbb{R}
\]
(f) \[
\begin{cases}
x_1 = 7 - 3t \\
x_2 = t \\
x_3 = 19 \\
x_4 = 3
\end{cases}; \quad t \in \mathbb{R}
\]
(g) \[
\begin{cases}
x_1 = -49 \\
x_2 = -24 \\
x_3 = 4 \\
x_4 = 1
\end{cases}
\]
8. (a) \[
\begin{align*}
x &= -5 \\
y &= 8 \\
z &= 3
\end{align*}
\] (b) \[
\begin{align*}
x &= 7 + 2s - 6t \\
y &= s \\
z &= t ; s, t \in \mathbb{R}
\end{align*}
\] (c) \[
\begin{align*}
x_1 &= 5 + 3t - 4s \\
x_2 &= s \\
x_3 &= -8 - 2t ; s, t \in \mathbb{R}
\end{align*}
\] (d) \[
\begin{align*}
x &= 1 - 6s - 4t \\
x_1 &= 8 + 3s - 7t \\
x_3 &= s \\
x_4 &= t \\
x_5 &= 2 \\
x_6 &= -10
\end{align*}
\] (e) \[
\begin{align*}
x_1 &= 1 - 3r + 2s - t \\
x_2 &= r \\
x_3 &= s \\
x_4 &= 2 + 8t \\
x_5 &= t
\end{align*}
\] 9. (a) \[
\begin{align*}
x &= 20 \\
y &= 9
\end{align*}
\] (b) \[
\begin{align*}
x &= -9/5 \\
y &= 26/5
\end{align*}
\] (c) \[
\begin{align*}
x &= -21 \\
y &= 13 \\
z &= 49
\end{align*}
\] (d) \[
\emptyset
\] (e) \[
\begin{align*}
x &= 23 - 16t \\
y &= 27 - 19t ; t \in \mathbb{R}
\end{align*}
\] (f) \[
\begin{align*}
x_1 &= -2 + 10t \\
x_2 &= 6 - 27t ; t \in \mathbb{R}
\end{align*}
\] (g) \[
\begin{align*}
x_1 &= 1 \\
x_2 &= -16 \\
x_3 &= 28 \\
x_4 &= 13
\end{align*}
\] (h) \[
\begin{align*}
x_1 &= 26 \\
x_2 &= 34 \\
x_3 &= -22 \\
x_4 &= -133
\end{align*}
\] 10. (a) \[
\begin{align*}
\emptyset : & \quad k = 1 \\
\infty : & \quad k = 2 \\
\emptyset : & \quad k = -3, -1 \\
\infty : & \quad k \in \mathbb{R} - \{-3, -1\}
\end{align*}
\] (c) \[
\begin{align*}
\emptyset : & \quad k = -4 \\
\infty : & \quad k = 4 \\
\emptyset : & \quad k = -8, 3, 5 \\
\infty : & \quad k \in \mathbb{R} - \{-8, 3, 5\}
\end{align*}
\] (d) \[
\begin{align*}
\emptyset : & \quad k = -3
\end{align*}
\] 11. (a) \[y = x^3 - x^2 + x - 1\] (b) The equation of a circle with center \((x_0, y_0)\) and radius \(r\) is naturally represented by the equation \((x - x_0)^2 + (y - y_0)^2 = r^2\). If we expand both squares and send the constant terms to the right side of the equality, the equation becomes \(x^2 + y^2 + (-2x_0)x + (-2y_0)y = r^2 - x_0^2 - y_0^2\). (c) \[x^2 + y^2 - 3x + y = 0\], with center \(\left(\frac{3}{2}, \frac{1}{2}\right)\) and radius \(\frac{\sqrt{10}}{2}\). 12. (a) \[
\begin{align*}
\theta &= 0, \pi, 2\pi \quad \text{, and so there are six solutions.}
\end{align*}
\] (b) \[
\begin{align*}
x &= \ln(3) \\
y &= -1/2, 1/2 \quad \text{, and so there are two solutions.}
\end{align*}
\] (c) \[
\begin{align*}
z &= -1/5
\end{align*}
\] 13.