Markov Chains

1. In each case, state whether the given matrix is a possible transition matrix for a Markov process.

(a) \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]
(b) \[
\begin{bmatrix}
0.4 & 0.3 \\
0.6 & 0.7
\end{bmatrix}
\]
(c) \[
\begin{bmatrix}
0.9 & -0.2 \\
0.1 & 1.2
\end{bmatrix}
\]
(d) \[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]
(e) \[
\begin{bmatrix}
0.5 & 0.5 \\
0.4 & 0.6
\end{bmatrix}
\]
(f) \[
\begin{bmatrix}
0.2 & 0.1 & 0.1 \\
0 & 0.3 & 0.7 \\
0.8 & 0 & 0.2
\end{bmatrix}
\]
(g) \[
\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

2. Consider the transition matrix \( P = \begin{bmatrix} 0.4 & 0 \\ 0.6 & 1 \end{bmatrix} \).

(a) Is \( P \) regular? Be sure to justify.
(b) If possible, find the steady-state vector of \( P \).
(c) If the steady-state vector exists, what does it imply about the Markov process?

3. Consider the transition matrix \( P = \begin{bmatrix} 1/4 & 0 & 2/3 \\ 0 & 1/2 & 1/3 \\ 3/4 & 1/2 & 0 \end{bmatrix} \).

(a) Is \( P \) regular? Be sure to justify.
(b) If possible, find the steady-state vector of \( P \).
(c) If the steady-state vector exists, use it to rank the states of the Markov process in decreasing order of likelihood.

4. A hotel employee has to clean the pool and hot tub every day. If on a particular day he cleans the pool first, then he will clean the pool first the following day 40% of the time. If on a particular day he cleans the hot tub first, then he will clean the hot tub first the following day 70% of the time.

(a) In the long run, what proportion of the time will the employee clean the pool first? the hot tub?
(b) Answer part (a) if 40% is changed for 50%.

5. An adventure video game is played in the jungle. Whether a level is sunny or cloudy is determined in the following way: if a level is sunny, the next level will be sunny 55% of the time while if a level is cloudy, the next level will be cloudy 60% of the time. In the long run, what proportion of levels will be sunny? cloudy?

6. A self-correcting robot performs a task on an assembly line. The task can be deemed excellent or average. After a week, data has shown that if a given task is deemed excellent, then the next one will be excellent as well 80% of the time. On the other hand, if a given task is deemed average, then the next one will be excellent 95% of the time. In the long run, what proportion of the tasks performed by the robot are excellent? average?

7. Consider a population formed by three neighbouring cities: city 1, 2 and 3. Every year, 5% and 8% of people living in city 1 move to city 2 and 3 respectively, 4% and 6% of people living in city 2 move to city 1 and 3 respectively and 3% and 2% of people living in city 3 move to city 1 and 2 respectively. Over time, what proportion of people are living in any given city?
Answers

1. (a) yes (b) yes (c) no (d) yes (e) no (f) no (g) yes

2. (a) No. $P$ is not regular since $P^n = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}$, for any integer $n \geq 1$.

(b) The steady-state vector $q$ exists and is equal to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(c) The steady-state vector implies that the Markov process will eventually be in state 2 and will remain in this state forever.

3. (a) Yes. $P$ is regular since $P^2 = \begin{bmatrix} 9/16 & 1/3 & 1/6 \\ 1/4 & 5/12 & 1/6 \\ 3/16 & 1/4 & 2/3 \end{bmatrix}$ has all strictly positive entries.

(b) The steady-state vector $q$ exists and is equal to $\begin{bmatrix} 8/23 \\ 6/23 \\ 9/23 \end{bmatrix}$.

(c) State 3, 1 and 2.

4. (a) The transition matrix is $P = \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix}$, with steady-state vector $q = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$.

This implies that the employee will clean the pool first $1/3$ of the time while he will clean the hot tub first $2/3$ of the time.

(b) The transition matrix is $P = \begin{bmatrix} 0.5 & 0.3 \\ 0.5 & 0.7 \end{bmatrix}$, with steady-state vector $q = \begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix}$.

This implies that the employee will clean the pool first $37.5\%$ of the time while he will clean the hot tub first $62.5\%$ of the time.

5. The transition matrix is $P = \begin{bmatrix} 0.55 & 0.4 \\ 0.45 & 0.6 \end{bmatrix}$, with steady-state vector $q = \begin{bmatrix} 8/17 \\ 9/17 \end{bmatrix}$.

This implies that about $47\%$ of the levels will be sunny while about $53\%$ of the levels will be cloudy.

6. The transition matrix is $P = \begin{bmatrix} 0.8 & 0.95 \\ 0.2 & 0.05 \end{bmatrix}$, with steady-state vector $q = \begin{bmatrix} 19/23 \\ 4/23 \end{bmatrix}$.

This implies that about $82.6\%$ of the tasks performed by the robot are deemed excellent while about $17.4\%$ are deemed average.

7. The transition matrix is $P = \begin{bmatrix} 0.87 & 0.04 & 0.03 \\ 0.05 & 0.9 & 0.02 \\ 0.08 & 0.06 & 0.95 \end{bmatrix}$, with steady-state vector $q = \begin{bmatrix} 134/619 \\ 123/619 \\ 362/619 \end{bmatrix}$.

This implies that over time, about $21.65\%$, $19.87\%$ and $58.48\%$ of the population is living in city 1, 2 and 3 respectively.