Gaussian Elimination

1. Start with the associated augmented matrix. Find the first column from the left containing a non-zero entry, say $\alpha$, and move the row containing $\alpha$ to the top position.

\[
\begin{bmatrix}
0 & 0 & * \\
0 & 0 & * \\
0 & \alpha & * \\
0 & * & *
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0 & \alpha & * \\
0 & 0 & * \\
0 & 0 & * \\
0 & * & *
\end{bmatrix}
\]

2. Divide the top row by $\alpha$ to create a leading $1$.

\[
\begin{bmatrix}
0 & 1 & * \\
0 & 0 & * \\
0 & 0 & * \\
0 & * & *
\end{bmatrix}
\]

3. Use the leading $1$ to kill all the entries below it in the same column.

\[
\begin{bmatrix}
0 & 1 & * \\
0 & 0 & * \\
0 & 0 & * \\
0 & 0 & *
\end{bmatrix}
\]

4. Ignore the top row and repeat step 1-2-3 on the matrix consisting of the remaining rows.

\[
\begin{bmatrix}
0 & 1 & * \\
0 & 0 & * \\
0 & 0 & *
\end{bmatrix}
\}
\text{Ignore.}
\]

\[
\begin{bmatrix}
0 & 0 & * \\
0 & 0 & *
\end{bmatrix}
\]
\text{Repeat step 1-2-3.}

This will eventually stop (since there are only finitely many rows to the augmented matrix).

Once this is complete, the matrix is said to be in row-echelon form.

The general solution can now be obtained using backward substitution.
Gauss-Jordan Elimination

1. Start with the associated augmented matrix. Find the first column from the left containing a non-zero entry, say $\alpha$, and move the row containing $\alpha$ to the top position.

\[
\begin{bmatrix}
0 & 0 & * \\
0 & 0 & * \\
0 & \alpha & * \\
0 & * & *
\end{bmatrix}
\begin{bmatrix}
0 & \alpha & * \\
0 & 0 & * \\
0 & 0 & * \\
0 & * & *
\end{bmatrix}
\]

2. Divide the top row by $\alpha$ to create a leading $1$.

\[
\begin{bmatrix}
0 & 1 & * \\
0 & 0 & * \\
0 & 0 & * \\
0 & * & *
\end{bmatrix}
\]

3. Use the leading $1$ to kill all the entries below it in the same column.

\[
\begin{bmatrix}
0 & 1 & * \\
0 & 0 & * \\
0 & 0 & * \\
0 & 0 & *
\end{bmatrix}
\]

4. Ignore the top row and repeat step 1-2-3 on the matrix consisting of the remaining rows.

\[
\begin{bmatrix}
0 & 1 & * \\
0 & 0 & * \\
0 & 0 & *
\end{bmatrix}
\]
\[
\begin{bmatrix}
0 & 0 & * \\
0 & 0 & *
\end{bmatrix}
\]

This will eventually stop (since there are only finitely many rows to the augmented matrix).

5. Starting with the last non-zero row and working upwards, introduce zeros above all the leading $1$’s. Once this is complete, the matrix is said to be in reduced row-echelon form.

The general solution can now be written without doing any addition work.