MATHEMATICS 201-NYC-05
Vectors and Matrices
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XVIII – Linear Transformations

1. Find the standard matrix for the linear transformation defined by the equations.

   a) $\begin{aligned} w_1 &= 2x_1 - 3x_2 + x_3 \\ w_2 &= 3x_1 + 5x_2 - x_3 \end{aligned}$

   b) $\begin{aligned} w_1 &= -x_1 + 3x_2 \\ w_2 &= 2x_2 \\ w_3 &= 4x_1 - 5x_2 \end{aligned}$

   c) $\begin{aligned} w_1 &= x_1 \\ w_2 &= x_1 - x_2 \\ w_3 &= x_1 + x_2 - x_3 \\ w_4 &= x_1 - x_2 + x_3 - x_4 \end{aligned}$

2. Find the standard matrix for the linear transformation T defined by the formula.

   a) $T(x_1, x_2) = (x_1 - x_2, x_2 - x_1, 2x_1 - 3x_2)$

   b) $T(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3)$

   c) $T(x_1, x_2, x_3) = (x_1, x_1 + 2x_2, 3x_1 + x_2, x_2)$

3. Find $T(X)$

   a) $[T] = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix}$ $\quad X = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

   b) $[T] = \begin{bmatrix} 2 & -1 \\ 3 & 2 \\ 4 & 5 \end{bmatrix}$ $\quad X = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

   c) $T(x_1, x_2, x_3) = (x_1 - x_2 + x_3, x_2 - x_3)$ $\quad X = (1, 2, 3)$

   d) $T(x_1, x_2, x_3, x_4) = (x_1, x_1 + x_2, x_1 + x_2 + x_3, x_1 + x_4, x_2 - x_3)$ $\quad X = (2, -1, -4, 3)$

4. Use a linear transformation to find the reflection of $(3, -7)$ across

   a) the x-axis b) the y-axis c) the line $y = x$

5. Use a linear transformation to find the orthogonal projection of $(3, -7)$ on

   a) the x-axis b) the y-axis

6. Use a linear transformation to find the image of the vector $(-3,5)$ when it is rotated through an angle of
   a) 30^{o} b) -60^{o} d) 90^{o}

7. Find the standard matrix for the stated composition of linear operators on $\mathbb{R}^2$
   a) A reflection across the line $y = x$ followed by a rotation of 90^{o}.
   b) A dilation with factor 3 followed by an orthogonal projection on the *x*-axis.
   c) A rotation of 60^{o} followed by a contraction with factor $\frac{1}{2}$ followed by a reflection across the *y*-axis.
   d) A rotation of 40^{o} followed by a rotation of 50^{o} followed by a rotation of 90^{o}.

8. Show that if the linear operator $T:\mathbb{R}^2 \to \mathbb{R}^2$ is a rotation, then *T* does not affect distances. That is, show that $\|(x,y)\| = \|T(x,y)\|$.

9. Determine whether the linear operator $T:\mathbb{R}^2 \to \mathbb{R}^2$ defined by the equations is one-to-one; if so, find the standard matrix for the inverse operator, and find $T^{-1}(w_1,w_2)$.
   a) $\begin{aligned} w_1 &= -x_1 + 3x_2 \\ w_2 &= 2x_1 + x_2 \end{aligned}$ b) $\begin{aligned} w_1 &= -x_1 + 3x_2 \\ w_2 &= 2x_1 - 6x_2 \end{aligned}$
   c) $\begin{aligned} w_1 &= x_2 \\ w_2 &= x_1 \end{aligned}$ d) $\begin{aligned} w_1 &= x_1 \\ w_2 &= 2x_1 \end{aligned}$

10. Determine whether the linear operator $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by the equations is one-to-one; if so, find the standard matrix for the inverse operator, and find $T^{-1}(w_1,w_2,w_3)$.
   a) $\begin{aligned} w_1 &= -x_1 + 2x_2 - 2x_3 \\ w_2 &= x_1 - 2x_2 + x_3 \\ w_3 &= 2x_1 + 4x_2 + x_3 \end{aligned}$ b) $\begin{aligned} w_1 &= -x_1 + 3x_2 - x_3 \\ w_2 &= 2x_2 + x_3 \\ w_3 &= 4x_1 - 5x_2 \end{aligned}$

11. Use the property of linear operators to determine whether $T:\mathbb{R}^2 \to \mathbb{R}^2$ is a linear operator.
   a) $T(x,y) = (2x,y)$ b) $T(x,y) = (x,y^2)$
   c) $T(x,y) = (-y,-x)$ d) $T(x,y) = \left(\sqrt[3]{x},\sqrt[3]{y}\right)$

12. Find the standard matrix for $T:\mathbb{R}^2 \to \mathbb{R}^2$ from the images of the standard basis vectors.
   a) *T* projects a vector orthogonally onto the *x*-axis and then reflects that vector about *y* the *y*-axis.
   b) *T* reflects a vector about the line $y = x$ and then reflects that vector about the *x*-axis.
   c) *T* dilates a vector by a factor of 3, then reflects that vector about the line $y = x$, and the projects that vector orthogonally onto the *y*-axis.
   d) *T* projects a vector onto the vector $\vec{v} = (3,-4)$.

13. For each of the following linear transformations $T(X) = AX$, find a basis for (i) range(T) and for (ii) ker(T).

a) $A=\begin{bmatrix}1 & -2\\ 2 & -4\end{bmatrix}$ b) $A=\begin{bmatrix}1 & 1 & 1\\ 2 & 2 & 3\end{bmatrix}$ c) $A=\begin{bmatrix}1 & -2 & 1\\ 2 & 2 & 2\\ 1 & -8 & 1\end{bmatrix}$

d) $A=\begin{bmatrix}-1 & 2 & 2 & 1 & 3\\ -2 & 4 & 3 & 2 & 0\\ 1 & -2 & -1 & -1 & 3\end{bmatrix}$ e) $T(x,y,z,w)=(x-y,x-y+3z+w,3x-3y+2w)$

14. For each of the following linear transformations $T(X) = AX$, find (i) range(T), (ii) rank(T), (iii) ker(T) and (iv) nullity(T).

a) $A=\begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix}$ b) $A=\begin{bmatrix}2 & -4\\ -4 & 8\end{bmatrix}$ c) $A=\begin{bmatrix}2 & -1\\ 3 & 1\\ 4 & 4\end{bmatrix}$

d) $A=\begin{bmatrix}1 & -2 & 4\\ 2 & 1 & 1\end{bmatrix}$ e) $A=\begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}$

Answers

1. a) $\begin{bmatrix} 2 & -3 & 1 \\ 3 & 5 & -1 \end{bmatrix}$ b) $\begin{bmatrix} -1 & 3 \\ 0 & 2 \\ 4 & -5 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & -1 \end{bmatrix}$

2. a) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 2 & -3 \end{bmatrix}$ b) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

3. a) $\begin{bmatrix} 8 \\ 9 \end{bmatrix}$ b) $\begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}$ c) $(2,-1)$ d) $(2,1,-3,5,3)$

4. a) $T\left(\begin{bmatrix} 3 \\ -7 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ -7 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ $\quad \therefore T(3,-7) = (3,7)$

   b) $T\left(\begin{bmatrix} 3 \\ -7 \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -7 \end{bmatrix} = \begin{bmatrix} -3 \\ -7 \end{bmatrix}$ $\quad \therefore T(3,-7) = (-3,-7)$

   c) $T\left(\begin{bmatrix} 3 \\ -7 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -7 \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \end{bmatrix}$ $\quad \therefore T(3,-7) = (-7,3)$

5. a) $T\left(\begin{bmatrix} 3 \\ -7 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -7 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ $\quad \therefore T(3,-7) = (3,0)$

   b) $T\left(\begin{bmatrix} 3 \\ -7 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -7 \end{bmatrix} = \begin{bmatrix} 0 \\ -7 \end{bmatrix}$ $\quad \therefore T(3,-7) = (0,-7)$

6. a) $T\left(\begin{bmatrix} -3 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{-3\sqrt{3}-5}{2} \\ \frac{-3+5\sqrt{3}}{2} \end{bmatrix}$ $\quad \therefore T(-3,5) = \left(\frac{-3\sqrt{3}-5}{2}, \frac{-3+5\sqrt{3}}{2}\right)$

   b) $T\left(\begin{bmatrix} -3 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{-3+5\sqrt{3}}{2} \\ \frac{3\sqrt{3}+5}{2} \end{bmatrix}$ $\quad \therefore T(-3,5) = \left(\frac{-3+5\sqrt{3}}{2}, \frac{3\sqrt{3}+5}{2}\right)$

   c) $T\left(\begin{bmatrix} -3 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$ $\quad \therefore T(-3,5) = (-5,-3)$

7. a) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} \frac{-1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}$ c) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

8. $\|T(x,y)\| = \|(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)\|$

$$= \sqrt{(x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2}$$

$$= \sqrt{x^2\cos^2\theta - 2xy\cos\theta\sin\theta + y^2\sin^2\theta + x^2\sin^2\theta + 2xy\sin\theta\cos\theta + y^2\cos^2\theta}$$

$$= \sqrt{x^2(\cos^2\theta + \sin^2\theta) + y^2(\sin^2\theta + \cos^2\theta)}$$

$$= \sqrt{x^2 + y^2}$$

$$= \|(x,y)\|$$

9. a) $T^{-1}(w_1, w_2) = \left(\frac{-1}{7}x_1 + \frac{3}{7}x_2, \frac{2}{7}x_1 + \frac{1}{7}x_2\right)$ b) not one-to-one
   c) $T^{-1}(w_1, w_2) = (x_2, x_1)$ d) not one-to-one

10. a) $T^{-1}(w_1, w_2, w_3) = \left(\frac{3}{4}x_1 + \frac{5}{4}x_2 + \frac{1}{4}x_3, \frac{-1}{8}x_1 - \frac{3}{8}x_2 + \frac{1}{8}x_3, -x_1 - x_2\right)$
    b) $T^{-1}(w_1, w_2, w_3) = \left(\frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3, \frac{4}{15}x_1 + \frac{4}{15}x_2 + \frac{1}{15}x_3, \frac{-8}{15}x_1 + \frac{7}{15}x_2 - \frac{2}{15}x_3\right)$

11. a) Linear Operator b) Not a linear operator
    c) Linear Operator d) Not a linear operator

12. a) $\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$ d) $\begin{bmatrix} \frac{9}{25} & \frac{-12}{25} \\ \frac{-12}{25} & \frac{16}{25} \end{bmatrix}$

13. a) $B_{\text{range}(T)} = \{(1,2)\}$ $B_{\ker(T)} = \{(2,1)\}$
    b) $B_{\text{range}(T)} = \{(1,2),(1,3)\}$ $B_{\ker(T)} = \{(-1,1,0)\}$
    c) $B_{\text{range}(T)} = \{(1,2,1),(-2,2,-8)\}$ $B_{\ker(T)} = \{(-1,0,1)\}$
    d) $B_{\text{range}(T)} = \{(-1,-2,1),(2,3,-1)\}$ $B_{\ker(T)} = \{(-9,0,-6,0,1),(1,0,0,1,0),(2,1,0,0,0)\}$
    e) $B_{\text{range}(T)} = \{(1,1,3),(0,3,0),(0,1,2)\}$ $B_{\ker(T)} = \{(1,1,0,0)\}$

14. a) $\text{range}(T) = \mathbb{R}^2$ $\text{rank}(T) = 2$ $\ker(T) = \{\vec{0}\}$ $\text{nullity}(T) = 0$
    b) $\text{range}(T) = \{t(2,-4) : t \in \mathbb{R}\}$ $\text{rank}(T) = 1$ $\ker(T) = \{t(2,1) : t \in \mathbb{R}\}$ $\text{nullity}(T) = 1$
    c) $\text{range}(T) = \{s(2,3,4) + t(-1,1,4) : s,t \in \mathbb{R}\}$ $\text{rank}(T) = 2$ $\ker(T) = \{\vec{0}\}$ $\text{nullity}(T) = 0$
    d) $\text{range}(T) = \mathbb{R}^2$ $\text{rank}(T) = 2$ $\ker(T) = \{t(-6,7,5)\}$ $\text{nullity}(T) = 1$
    e) $\text{range}(T) = \mathbb{R}^2$ $\text{rank}(T) = 2$
       $\ker(T) = \{s(0,0,-1,1) + t(-1,1,0,0) : s,t \in \mathbb{R}\}$ $\text{nullity}(T) = 2$