**MATHEMATICS 201-NYC-05**
Vectors and Matrices
Martin Huard
Fall 2007

# XVII – Rank and Nullity

1. For each of the matrices, find
   i) a basis for the row space
   ii) a basis for the column space
   iii) the rank
   iv) a basis for the nullspace
   v) the nullity

   a) $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ b) $\begin{bmatrix} 2 & -6 \\ 1 & 4 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix}$

   d) $\begin{bmatrix} 1 & -2 & 3 \\ 4 & 1 & 3 \\ -1 & -7 & 6 \end{bmatrix}$ e) $\begin{bmatrix} 1 & -3 & 2 \\ 3 & 1 & 5 \\ 2 & -4 & 3 \end{bmatrix}$ f) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

   g) $\begin{bmatrix} 1 & -2 & 4 & 1 \\ 3 & -6 & 5 & -9 \end{bmatrix}$ h) $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 3 & 6 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ i) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2. Find a basis for the subspace spanned by $S$. For those vectors not in the basis, express them as a linear combination of the basis vectors.
   a) $S = \{(2,-1,4),(3,2,5),(1,-4,3)\}$
   b) $S = \{(1,-2,2),(-3,6,-6),(3,1,1)\}$
   c) $S = \{(2,-1,4),(4,-2,1),(-6,3,5)\}$
   d) $S = \{(1,-1,4,1),(4,-3,3,4),(2,-1,-5,2),(1,-2,17,1)\}$

3. Find a subset of the vectors in $S$ that form a basis for subspace spanned by $S$. For those vectors not in the basis, express them as a linear combination of the basis vectors.
   a) $S = \{(2,-1,4),(3,2,5),(1,-4,3)\}$
   b) $S = \{(1,-2,2),(-3,6,-6),(3,1,1)\}$
   c) $S = \{(1,-3,5),(3,10,2),(1,-22,18)\}$
   d) $S = \{(1,-1,4,1),(4,-3,3,4),(2,-1,-5,2),(1,-2,17,1)\}$

4. Find a basis and the dimension for the solution space of $AX=0$.

a) $\begin{aligned} x + y + z &= 0 \\ -2x - y + 2z &= 0 \\ -x + 3z &= 0 \end{aligned}$

b) $\begin{aligned} 3x + y + z + w &= 0 \\ 5x - y + z - w &= 0 \end{aligned}$

c) $\begin{aligned} x - 2y + 3z &= 0 \\ 2x - 4y + 6z &= 0 \\ -x + 2y - 3z &= 0 \end{aligned}$

d) $\begin{aligned} x - 2y + z &= 0 \\ 3x - 6y + z &= 0 \\ -x + 2y + 3z &= 0 \\ 5x - 10y - z &= 0 \end{aligned}$

e) $3x + 2y - 5z = 0$

5. Write the solution to the system $AX=b$ in the form $X = X_p + X_h$, where $X_h$ is a solution of $AX=0$ and $X_p$ is a particular solution of $AX=b$.

a) $\begin{aligned} x + 3y + 10z &= 18 \\ -2x + 7y + 32z &= 29 \\ -x + 3y + 14z &= 12 \\ x + y + 2z &= 8 \end{aligned}$

b) $\begin{aligned} 2x + 3y - z + w &= 3 \\ x - y + 2w &= -2 \\ 4x + y - z + 5w &= -1 \end{aligned}$

c) $\begin{aligned} 2x - 3y + z &= 13 \\ 5x - 2y + 2z &= 21 \\ 3x + y + 2z &= 9 \end{aligned}$

6. Find the largest possible value for the rank of *A* and the smallest possible value for the nullity of *A*.

a) $A_{4\times 4}$ b) $A_{2\times 7}$ c) $A_{7\times 2}$

7. Are there any values for *s* and *t* such that the following matrix has rank 1 or 2? Find those values.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & t+1 & t-1 \\ 0 & s-5 & s+5 \\ 0 & 0 & 2 \end{bmatrix}$$

8. Determine if the system of linear equation $AX=b$ is consistent using the given information. If so, determine the number of parameters in the general solution.

| | Size of *A* | $Rank(A)$ | $Rank([A\mid b])$ |
|---|---|---|---|
| a) | $3\times 3$ | 3 | 3 |
| b) | $3\times 3$ | 2 | 3 |
| c) | $3\times 3$ | 2 | 2 |
| d) | $4\times 9$ | 3 | 4 |
| e) | $4\times 9$ | 2 | 2 |

# Answers

1. a) $B_{RS}=\{(1,0),(0,1)\}$, $B_{CS}=\{(1,0),(0,3)\}$, rank = 2, $B_{NS}=\varnothing$, nullity = 0
   b) $B_{RS}=\{(1,-3),(0,1)\}$, $B_{CS}=\{(2,1),(-6,4)\}$, rank = 2, $B_{NS}=\varnothing$, nullity = 0
   c) $B_{RS}=\{(1,5)\}$, $B_{CS}=\{(1,2)\}$, rank = 1, $B_{NS}=\{(-5,1)\}$, nullity = 1
   d) $B_{RS}=\{(1,-2,3),(0,1,-1)\}$, $B_{CS}=\{(1,4,-1),(-2,1,-7)\}$, rank = 2, $B_{NS}=\{(-1,1,1)\}$, nullity = 1
   e) $B_{RS}=\{(1,-3,2),(0,1,\tfrac{-1}{10}),(0,0,1)\}$, $B_{CS}=\{(1,3,2),(-3,1,-4),(2,5,3)\}$, rank = 3, $B_{NS}=\varnothing$, nullity = 0
   f) $B_{RS}=\{(1,2,3)\}$, $B_{CS}=\{(1)\}$, rank = 1, $B_{NS}=\{(-2,1,0),(-3,0,1)\}$, nullity = 2
   g) $B_{RS}=\{(1,-2,4,1),(0,0,1,\tfrac{12}{7})\}$, $B_{CS}=\{(1,3),(4,5)\}$, rank = 2, $B_{NS}=\{(2,1,0,0),(\tfrac{41}{7},0,\tfrac{-12}{7},1)\}$, nullity = 2
   h) $B_{RS}=\{(1,2,1),(0,0,1)\}$, $B_{CS}=\{(1,2,3,1),(2,4,1,1)\}$, rank = 2, $B_{NS}=\{(-2,1,0)\}$, nullity = 1
   i) $RS=CS=\{\vec{0}\}$ (no basis), rank = 0, $B_{NS}=\{(1,0),(0,1)\}$, nullity = 2
2. a) $B=\{(1,\tfrac{-1}{2},2),(0,1,\tfrac{-2}{7})\}$ b) $B=\{(1,-2,2),(0,1,\tfrac{-5}{7})\}$
   c) $B=\{(1,\tfrac{-1}{2},2),(0,0,1)\}$ d) $B=\{(1,-1,4,1),(0,1,-13,0)\}$
3. a) $B=\{(2,-1,4),(3,2,5)\}$ $(1,-4,3)=2(2,-1,4)-(3,2,5)$
   b) $S=\{(1,-2,2),(3,1,1)\}$ $(-3,6,-6)=-3(1,-2,2)$
   c) $B=\{(1,-3,5),(3,10,2)\}$ $(1,-22,18)=4(1,-3,5)-(3,10,2)$
   d) $B=\{(1,-1,4,1),(4,-3,3,4)\}$ $(2,-1,-5,2)=-2(1,-1,4,1)+(4,-3,3,4)$
   $(1,-2,17,1)=5(1,-1,4,1)-(4,-3,3,4)$
4. a) $\{(3,-4,1)\}$ b) $\{(-1,-1,4,0),(0,-1,0,1)\}$ c) $\{(2,1,0),(1,0,1)\}$
   d) $\{(2,1,0)\}$ e) $\{(-2,3,0),(5,0,3)\}$
5. a) $X_h=t(2,-4,1)$ $X_p=(3,5,0)$ $X=t(2,-4,1)+(3,5,0)$
   b) $X_h=s(\tfrac{1}{5},\tfrac{1}{5},1,0)+t(\tfrac{-7}{5},\tfrac{3}{5},0,1)$ $X_p=(\tfrac{-3}{5},\tfrac{7}{5},0,0)$
   $X=s(\tfrac{1}{5},\tfrac{1}{5},1,0)+t(\tfrac{-7}{5},\tfrac{3}{5},0,1)+(\tfrac{-3}{5},\tfrac{7}{5},0,0)$
   c) $X_h=(0,0,0)$ $X_p=(3,-2,1)$ $X=(3,-2,1)$
6. a) 4, 0 b) 2, 5 c) 2, 0
7. Rank 2 when $t=-1$ and $s=5$. It can never have rank 1.
8. a) Yes, 0 b) No c) Yes, 1 d) No e) Yes, 7