MATHEMATICS 201-NYC-05
Vectors and Matrices
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XVI - Basis and Dimension

1. Explain why S is not a basis for $\mathbb{R}^2$. Give which of the two conditions fails.
   a) $S=\{(3,1),(-1,2),(4,1)\}$
   b) $S=\{(3,-5),(-6,10)\}$

2. Explain why S is not a basis for $\mathbb{R}^3$. Give which of the two conditions fails.
   a) $S=\{(1,3,-2),(2,4,1),(5,11,0)\}$
   b) $S=\{(2,3,-5),(1,-6,10)\}$
   c) $S=\{(1,2,3),(5,1,-2),(3,-4,5),(1,-1,1)\}$

3. Explain why S is not a basis for P_2. Give which of the two conditions fails.
   a) $S=\{x^2-1,2x+1,3x^2+6x\}$
   b) $S=\{1+x,5\}$
   c) $S=\{2x^2-3x+1,x^2+x-1,3x^2+2x,x^2+6x-4\}$

4. Explain why S is not a basis for $M_{2,2}$. Give which of the two conditions fails.
   a) $S=\left\{\begin{bmatrix}1&0\\0&2\end{bmatrix},\begin{bmatrix}0&1\\2&0\end{bmatrix}\right\}$
   b) $S=\left\{\begin{bmatrix}1&2\\0&0\end{bmatrix},\begin{bmatrix}0&0\\2&1\end{bmatrix},\begin{bmatrix}1&2\\2&1\end{bmatrix},\begin{bmatrix}1&2\\3&4\end{bmatrix}\right\}$

5. Determine whether the set S is a basis for the indicated vector space. Verify both conditions!
   a) $S=\{(2,5),(-2,5)\}$ for $\mathbb{R}^2$
   b) $S=\{(2,1,0),(1,0,2),(3,1,2)\}$ for $\mathbb{R}^3$
   c) $S=\{(1,2,3,4),(0,1,2,3),(0,0,1,2),(0,0,0,1)\}$ for $\mathbb{R}^4$
   d) $S=\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}1&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&1\end{bmatrix}\right\}$ for $M_{2,2}$
   e) $S=\{1+x+x^2,1+3x,x^2-x-5\}$ for P_2

6. Find a basis for $D_{3,3}$ (the vector space of all 3×3 diagonal matrices). What is the dimension of this vector space?

7. Find all subsets of the following set that form a basis for $\mathbb{R}^2$.
   $S=\{(1,0),(0,1),(1,1),(2,2)\}$

8. Find the coordinate vector of $\vec{w}$ relative to the basis S of V.

a) $\vec{w} = (3,19,2)$ $S = \{(8,11,0),(7,0,10),(1,4,6)\}$ $V = \mathbb{R}^3$

b) $\vec{w} = (5,-12,3)$ $S = \{(1,2,3),(-4,5,6),(7,-8,9)\}$ $V = \mathbb{R}^3$

c) $\vec{w} = (11,18,-7)$ $S = \{(4,3,3),(-11,0,11),(0,9,2)\}$ $V = Span(S)$

d) $\vec{w} = (-1,-5,-2)$ $S = \{(3,-1,4),(5,1,7)\}$ $V = Span(S)$

e) $\vec{w} = (6,-12,18)$ $S = \{(-1,2,-3)\}$ $V = Span(S)$

9. Find the coordinate vector of $p(x)$ relative to the basis S of V.

a) $p(x) = 2 - x + x^2$ $S = \{1+x, 1+x^2, x+x^2\}$ $V = P_2$

b) $p(x) = 3x^2 - 8x - 21$ $S = \{3x^2 - 5, 2x + 4\}$ $V = Span(S)$

10. Find the coordinate vector of $A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$ relative to the basis

$$S = \left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

11. Consider the subspace W of $\mathbb{R}^2$ given by $W = \{(t,3t) : t \in \mathbb{R}\}$.

a) Find a basis for W.

b) Determine the dimension of W.

c) Give a description of W.

12. Consider the subspace W of $\mathbb{R}^3$ given by $W = \{(t,-t,3t) : t \in \mathbb{R}\}$.

a) Find a basis for W.

b) Determine the dimension of W.

c) Give a description of W.

13. Consider the subspace W of $\mathbb{R}^3$ given by $W = \{(t+s,t,s) : t,s \in \mathbb{R}\}$.

a) Find a basis for W.

b) Determine the dimension of W.

c) Give a description of W.

14. Consider the subspace W of $\mathbb{R}^3$ given by $W = \{(0,t+s,2s) : t,s \in \mathbb{R}\}$.

a) Find a basis for W.

b) Determine the dimension of W.

c) Give a description of W.

15. Consider the subspace W of $\mathbb{R}^3$ given by $W = \left\{\vec{u} \in \mathbb{R}^3 : \vec{u} \perp (2,-1,5)\right\}$.
    a) Find a basis for W.
    b) Determine the dimension of W.
    c) Give a description of W.

16. Consider the subspace W of $\mathbb{R}^5$ given by
    $W = \left\{\vec{u} \in \mathbb{R}^5 : \vec{u} \perp (2,-1,5,2,2) \text{ and } \vec{u} \perp (3,-1,2,0,1)\right\}$.
    a) Find a basis for W.
    b) Determine the dimension of W.

17. Consider the subspace W of $M_{2,2}$ given by $W = \left\{\begin{bmatrix} t & t+s \\ t+s & s \end{bmatrix} : t, s \in \mathbb{R}\right\}$.
    a) Find a basis for W.
    b) Determine the dimension of W.

18. Do the following sets W form a basis for the vector space V? Use the dimension of V to facilitate your work.
    a) $W = \left\{(2,-3,4,1), (1,2,3,-8), (2,1,-5,4)\right\}$ $\quad V = \mathbb{R}^4$
    b) $W = \left\{x^2 + x + 1, x + 1\right\}$ $\quad V = P_2$
    c) $W = \left\{(2,-3,4), (1,2,3), (2,1,-5)\right\}$ $\quad V = \mathbb{R}^3$
    d) $W = \left\{(2,-3,4), (1,2,3), (3,-8,5)\right\}$ $\quad V = \mathbb{R}^3$
    e) $W = \left\{(1,-3,4), (1,-5,5), (1,1,1), (2,-4,5)\right\}$ $\quad V = \mathbb{R}^3$
    f) $W = \left\{\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}\right\}$ $\quad V = M_{2,2}$

19. Extend W to form a basis for $\mathbb{R}^3$ by adding a vector $\vec{w}$ to the set.
    a) $W = \left\{(1,2,-1), (3,1,0)\right\}$
    b) $W = \left\{(1,1,1), (1,0,1)\right\}$

20. Do the following sets W form a basis for $\mathbb{R}^3$? If not, add or subtract vectors from the set to form a basis.
    a) $W = \left\{(1,2,3)\right\}$
    b) $W = \left\{(1,2,3), (4,5,6)\right\}$
    c) $W = \left\{(1,2,3), (4,5,6), (7,8,9)\right\}$
    d) $W = \left\{(1,-2,0), (0,3,4), (1,1,1)\right\}$
    e) $W = \left\{(1,-2,3), (2,-2,3), (2,-4,6), (1,-4,6)\right\}$

21. Find a basis for W and give a geometrical description for a, b and c.

a) $W=\left\{(a+b-c,2a+2b-c,-a-b-c):a,b,c\in\mathbb{R}\right\}$

b) $W=\left\{(a+b+c,2a-4b+c):a,b,c\in\mathbb{R}\right\}$

c) $W=\left\{(2a-b+5c,a+4b-2c,3a+b+5c)\right\}$

d) $W=\left\{\begin{bmatrix} a+b-c & a-b-c \\ 2a+b-2c & 2a-b-2c \end{bmatrix}:a,b,c\in\mathbb{R}\right\}$

22. Find the basis for $W=\left\{p(x)\in P_2 : p(1)=0\right\}$.

23. Find the basis for $W=\left\{p(x)\in P_2 : p'(1)=0\right\}$

24. For each of the given sets A and B in $\mathbb{R}^3$,

i) Find a basis for $span(A)$ and give a geometrical description.

ii) Find a basis for $span(B)$ and give a geometrical description.

iii) Find a basis for $span(A)+span(B)$ and give a geometrical description.

iv) Find a basis for $span(A)\cap span(B)$ and give a geometrical description.

a) $A=\left\{(1,-2,4),(1,1,1)\right\}$ $B=\left\{(2,-1,3)\right\}$

b) $A=\left\{(1,2,3),(4,5,6),(5,6,7)\right\}$ $B=\left\{(1,1,1),(2,2,2),(3,3,3)\right\}$

c) $A=\left\{(1,-3,4),(3,-1,5),(7,3,7)\right\}$ $B=\left\{(2,-1,3),(1,1,-4),(1,-2,7)\right\}$

d) $A=\left\{(1,0,2),(1,-2,4),(1,1,1)\right\}$ $B=\left\{(1,-5,7),(2,-1,5),(0,1,-1)\right\}$

e) $A=\left\{(1,-2,3),(3,1,0),(2,0,1)\right\}$ $B=\left\{(3,1,-4),(1,3,-2),(-7,3,8),(5,-1,-6)\right\}$

Answers

1. a) S is linearly dependent b) S does not span $\mathbb{R}^2$ and S is linearly dependent
2. a) S does not span $\mathbb{R}^3$ and S is linearly dependent
   b) S does not span $\mathbb{R}^3$ c) S is linearly dependent
3. a) S does not span P_2 and S is linearly dependent
   b) S does not span P_2 c) S does not span P_2 and S is linearly dependent
4. a) S does not span $M_{2,2}$ b) S does not span $M_{2,2}$ and S is linearly dependent
5. a) Yes b) No c) Yes d) No e) Yes
6. $S=\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\}$ $\dim\left(D_{3,3}\right)=3$
7. $W_1=\{(1,0),(0,1)\}$, $W_2=\{(1,0),(1,1)\}$, $W_3=\{(1,1),(0,1)\}$, $W_4=\{(1,0),(2,2)\}$ and
   $W_5=\{(0,1),(2,2)\}$
8. a) $(\vec{w})_S=(1,-1,2)$ b) $(\vec{w})_S=(-2,0,1)$ c) $(\vec{w})_S=(0,-1,2)$
   d) $(\vec{w})_S=(3,-2)$ e) $(\vec{w})_S=(-6)$
9. a) $(p(x))_S=(0,2,-1)$ b) $(p(x))_S=(1,-4)$
10. $(A)_S=(-1,1,-1,3)$
11. a) $S=\{(1,3)\}$ b) $\dim(W)=1$ c) The line $y=3x$
12. a) $S=\{(1,-1,3)\}$ b) $\dim(W)=1$ c) The line $x=-y=\frac{z}{3}$
13. a) $S=\{(1,1,0),(1,0,1)\}$ b) $\dim(W)=2$ c) The plane $x-y-z=0$
14. a) $S=\{(0,1,0),(0,1,2)\}$ b) $\dim(W)=2$ c) The plane $x=0$
15. a) $B=\{(1,2,0),(5,0,-2)\}$ b) $\dim(W)=2$ c) The plane $2x-y+5z=0$
16. a) $B=\{(3,11,1,0,0),(2,6,0,1,0),(1,4,0,0,1)\}$ b) $\dim(W)=3$
17. a) $S=\left\{\begin{bmatrix}1&1\\1&0\end{bmatrix},\begin{bmatrix}0&1\\1&1\end{bmatrix}\right\}$ b) $\dim(W)=2$
18. a) No b) No c) Yes d) No e) No f) No
19. a) $\vec{w}=(1,0,0)$ b) $\vec{w}=(1,0,0)$
20. a) No. $W'=\{(1,2,3),(1,0,0),(0,1,0)\}$ b) No $W'=\{(1,2,3),(4,5,6),(1,0,0)\}$
    c) No. $W'=\{(1,2,3),(4,5,6),(1,0,0)\}$ d) Yes
    e) No. $W'=\{(1,-2,3),(2,-2,3),(0,0,1)\}$
21. a) $B=\{(1,2,-1),(-1,-1,-1)\}$ The plane $x-2y-z=0$
    b) $B=\{(1,2),(1,-4)\}$ $\mathbb{R}^2$

c) $B=\{(2,1,3),(-1,4,1)\}$ The plane $11x-5y+9z=0$

d) $B=\left\{\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix},\begin{bmatrix}1 & -1\\ 1 & -1\end{bmatrix}\right\}$

22. $B=\{-x^2+1,-x^2+x\}$

23. $B=\{\frac{-1}{2}x^2+x,1\}$

24. a) $B_{span(A)}=\{(1,-2,4),(1,1,1)\}$ and $span(A)$ describes the plane $2x-y-z=0$.

$B_{span(B)}=\{(2,-1,3)\}$ and $span(B)$ describes the line $\frac{x}{2}=-y=\frac{z}{3}$.

$B_{span(A)+span(B)}=\{(1,-2,4),(1,1,1),(2,-1,3)\}$ and describes $\mathbb{R}^3$.

$span(A)\cap span(B)$ has no basis and describes the origin.

b) $B_{span(A)}=\{(1,2,3),(4,5,6)\}$ and $span(A)$ describes the plane $x-2y+z=0$.

$B_{span(B)}=\{(1,1,1)\}$ and $span(B)$ describes the line $x=y=z$.

$B_{span(A)}=\{(1,2,3),(4,5,6)\}$ and describes the plane $x-2y+z=0$

$span(A)\cap span(B)$ has no basis and describes the origin.

c) $B_{span(A)}=\{(1,-3,4),(3,-1,5)\}$ and $span(A)$ describes the plane $11x-7y-8z=0$

$B_{span(B)}=\{(2,-1,3),(1,1,-4)\}$ and $span(B)$ describes the plane $x+11y+3z=0$

$B_{span(A)+span(B)}=\{(1,-3,4),(3,-1,5),(2,-1,3)\}$ and describes $\mathbb{R}^3$.

$B_{span(A)\cap span(B)}=\{(2,1,3)\}$ and describes the line $\frac{128}{67}x=-\frac{128}{41}y=z$.

d) $B_{span(A)}=\{(1,0,2),(1,-2,4)\}$ and $span(A)$ describes the plane $2x-y-z=0$.

$B_{span(B)}=\{(1,-5,7),(2,-1,5)\}$ and $span(B)$ describes the line $2x-y-z=0$.

$B_{span(A)+span(B)}=\{(1,-2,4),(1,1,1)\}$ and describes the plane $2x-y-z=0$

$B_{span(A)\cap span(B)}=\{(1,-2,4),(1,1,1)\}$ and describes the plane $2x-y-z=0$.

e) $B_{span(A)}=\{(1,-2,3),(3,1,0),(2,0,1)\}$ and $span(A)$ describes $\mathbb{R}^3$.

$B_{span(B)}=\{(3,1,-4),(1,3,-2)\}$ and $span(B)$ describes the plane $5x+y+4z=0$.

$B_{span(A)+span(B)}=\{(1,-2,3),(3,1,0),(2,0,1)\}$ and describes $\mathbb{R}^3$.

$B_{span(A)\cap span(B)}=\{(3,1,-4),(1,3,-2)\}$ and describes the plane $5x+y+4z=0$.