MATHEMATICS 201-NYC-05
Vectors and Matrices
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XIV - Vector Spaces and Subspaces

1. Describe the zero vector (the additive identity) for the following vector spaces.
   a) $\mathbb{R}^4$ b) $C(-\infty,\infty)$ c) $M_{2,3}$ d) P_3
   e) $V=\{(x,y): x,y\in\mathbb{R},\ x>0\}$ with the following operations :
   $$(x_1,y_1)\oplus(x_2,y_2)=(x_1x_2,y_1+y_2)$$
   $$k\odot(x_1,y_1)=(x_1^{\,k},ky_1)$$

2. Describe the additive inverse of a vector for the following vector spaces.
   a) $\mathbb{R}^4$ b) $C(-\infty,\infty)$ c) $M_{2,3}$ d) P_3
   e) $V=\{(x,y): x,y\in\mathbb{R},\ x>0\}$ with the following operations :
   $$(x_1,y_1)\oplus(x_2,y_2)=(x_1x_2,y_1+y_2)$$
   $$k\odot(x_1,y_1)=(x_1^{\,k},ky_1)$$

3. Determine whether the given set, together with the indicated operations, is a vector space. If it is, prove that each axiom is satisfied, if it is not, identify the axioms that fail.
   a) $M_{2,3}$ with standard operation
   b) $\mathbb{R}^3$ with standard operation
   c) P_3 with the standard operation
   d) The set $\{(x,y): x\ge 0, y\ge 0\}$ with standard operations
   e) The set of all 2×2 matrices of the form $\begin{bmatrix} a & 1 \\ 1 & b\end{bmatrix}$ with standard operations
   f) The set $\{ax^5 : a\in\mathbb{R}\}$.
   g) $\mathbb{R}^2$ with the following operations : $(x_1,y_1)\oplus(x_2,y_2)=(x_1+x_2,y_1+y_2)$
   $$k\odot(x_1,y_1)=(kx_1,y_1)$$
   h) $\mathbb{R}^2$ with the following operations : $(x_1,y_1)\oplus(x_2,y_2)=(x_1,0)$
   $$k\odot(x_1,y_1)=(kx_1,ky_1)$$
   i) $\mathbb{R}^2$ with the following operations : $(x_1,y_1)\oplus(x_2,y_2)=(x_1x_2,y_1y_2)$
   $$k\odot(x_1,y_1)=(kx_1,ky_1)$$
   j) $\mathbb{R}^2$ with the following operations : $(x_1,y_1)\oplus(x_2,y_2)=(x_1+x_2,y_1+y_2)$
   $$k\odot(x_1,y_1)=(k^2x_1,k^2y_1)$$

4. Consider the set V whose only element is moon, that is, $V = \{\text{moon}\}$. Is this set a vector space under the following operations?

   moon $\oplus$ moon = moon

   $k \odot$(moon) = moon for every real number k

5. Determine whether the subset W of $\mathbb{R}^3$, with the standard operations, is a vector space. Justify your answer. (*Hint*: Show that W is a subspace of $\mathbb{R}^3$).
   a) $W = \{(a,b,0) : a,b \in \mathbb{R}\}$
   b) $W = \{(a,1,1) : a \in \mathbb{R}\}$
   c) $W = \{(a,b,a+b) : a,b \in \mathbb{R}\}$
   d) $W = \{(a,b,ab) : a,b \in \mathbb{R}\}$
   e) $W = \{(a,b-a,b) : a,b \in \mathbb{R}\}$
   f) $W = \{(x,y,z) : x-2y+z=0\}$
   g) $W = \{(x,y,z) : 2x+y-z-3=0\}$
   h) $W = \{(x,y,z) : x=2t, y=-t, z=5t,\ t \in \mathbb{R}\}$
   i) $W = \{u \in \mathbb{R}^3 : \vec{u} \perp \vec{w} = (2,-1,5)\}$

6. Determine whether the subset W of $M_{2,2}$ with the standard operations is a vector space. Justify your answer.
   a) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{Z} \right\}$
   b) W is the set of 2×2 matrices A such that $\det(A) = 0$
   c) W is the set of 2×2 symmetric matrices A
   d) W is the set of diagonal 2×2 matrices.
   e) $W = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$
   f) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a+b+c+d = 0 \right\}$

7. Determine whether the subset W of $C(-\infty,\infty)$ is a subspace of $C(-\infty,\infty)$. Justify your answer.
   a) The set of nonnegative functions: $f(x) \geq 0$
   b) The set of all even functions: $f(-x) = f(x)$
   c) The set of all odd functions: $f(-x) = -f(x)$
   d) The set of all constant functions: $f(x) = c,\ c \in \mathbb{R}$.
   e) The set of all functions such that $f(0) = 0$
   f) The set of all functions such that $f(0) = 1$

ANSWERS

1. a) $(0,0,0,0)$ b) $f(x)=0$ c) $\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}$ d) $p(x)=0+0x+0x^2+0x^3$

   e) (1,0)

2. a) $\vec{u}=(u_1,u_2,u_3,u_4)$
   $-\vec{u}=(-u_1,-u_2,-u_3,-u_4)$

   b) $(-f)(x)=-f(x)$

   c) $A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{bmatrix}$
   $-A=\begin{bmatrix}-a_{11}&-a_{12}&-a_{13}\\-a_{21}&-a_{22}&-a_{23}\end{bmatrix}$

   d) $p(x)=a_0+a_1x+a_2x^2+a_3x^3$
   $-p(x)=-a_0-a_1x-a_2x^2-a_3x^3$

   e) $\vec{u}=(x_1,y_1)$
   $-\vec{u}=\left(\frac{1}{x_1},-y_1\right)$

3. a) Let $A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{bmatrix}$, $B=\begin{bmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\end{bmatrix}$ and $C=\begin{bmatrix}c_{11}&c_{12}&c_{13}\\c_{21}&c_{22}&c_{23}\end{bmatrix}$

   1. $A+B=\begin{bmatrix}a_{11}+b_{11}&a_{12}+b_{12}&a_{13}+b_{13}\\a_{21}+b_{21}&a_{22}+b_{22}&a_{23}+b_{23}\end{bmatrix}$ is a 2×3 matrix

   2. $A+B=\begin{bmatrix}a_{11}+b_{11}&a_{12}+b_{12}&a_{13}+b_{13}\\a_{21}+b_{21}&a_{22}+b_{22}&a_{23}+b_{23}\end{bmatrix}=\begin{bmatrix}b_{11}+a_{11}&b_{12}+a_{12}&b_{13}+a_{13}\\b_{21}+a_{21}&b_{22}+a_{22}&b_{23}+a_{23}\end{bmatrix}=B+A$

   3. $$A+(B+C)=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{bmatrix}+\begin{bmatrix}b_{11}+c_{11}&b_{12}+c_{12}&b_{13}+c_{13}\\b_{21}+c_{21}&b_{22}+c_{22}&b_{23}+c_{23}\end{bmatrix}$$
   $$=\begin{bmatrix}a_{11}+(b_{11}+c_{11})&a_{12}+(b_{12}+c_{12})&a_{13}+(b_{13}+c_{13})\\a_{21}+(b_{21}+c_{21})&a_{22}+(b_{22}+c_{22})&a_{23}+(b_{23}+c_{23})\end{bmatrix}$$
   $$=\begin{bmatrix}(a_{11}+b_{11})+c_{11}&(a_{12}+b_{12})+c_{12}&(a_{13}+b_{13})+c_{13}\\(a_{21}+b_{21})+c_{21}&(a_{22}+b_{22})+c_{22}&(a_{23}+b_{23})+c_{23}\end{bmatrix}$$
   $$=\begin{bmatrix}a_{11}+b_{11}&a_{12}+b_{12}&a_{13}+b_{13}\\a_{21}+b_{21}&a_{22}+b_{22}&a_{23}+b_{23}\end{bmatrix}+\begin{bmatrix}c_{11}&c_{12}&c_{13}\\c_{21}&c_{22}&c_{23}\end{bmatrix}$$
   $$=(A+B)+C$$

   4. $$A+0_{2\times 3}=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{bmatrix}+\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}$$
   $$=\begin{bmatrix}a_{11}+0&a_{12}+0&a_{13}+0\\a_{21}+0&a_{22}+0&a_{23}+0\end{bmatrix}=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{bmatrix}=A$$

5. $$A+(-A)=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}+\begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \end{bmatrix}$$
$$=\begin{bmatrix} a_{11}-a_{11} & a_{12}-a_{12} & a_{13}-a_{13} \\ a_{21}-a_{21} & a_{22}-a_{22} & a_{23}-a_{23} \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$=0_{2\times 3}$$

6. $kA=\begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \end{bmatrix}$ is a 2×3 matrix

7. $$k(A+B)=\begin{bmatrix} k(a_{11}+b_{11}) & k(a_{12}+b_{12}) & k(a_{13}+b_{13}) \\ k(a_{21}+b_{21}) & k(a_{22}+b_{22}) & k(a_{23}+b_{23}) \end{bmatrix}$$
$$=\begin{bmatrix} ka_{11}+kb_{11} & ka_{12}+kb_{12} & ka_{13}+kb_{13} \\ ka_{21}+kb_{21} & ka_{22}+ka_{22} & ka_{23}+ka_{23} \end{bmatrix}$$
$$=kA+kB$$

8. $$(k+l)A=\begin{bmatrix} (k+l)a_{11} & (k+l)a_{12} & (k+l)a_{13} \\ (k+l)a_{21} & (k+l)a_{22} & (k+l)a_{23} \end{bmatrix}$$
$$=\begin{bmatrix} ka_{11}+la_{11} & ka_{12}+la_{12} & ka_{13}+la_{13} \\ ka_{21}+la_{21} & ka_{22}+la_{22} & ka_{23}+la_{23} \end{bmatrix}$$
$$=kA+lA$$

9. $$k(lA)=\begin{bmatrix} k(la_{11}) & k(la_{12}) & k(la_{13}) \\ k(la_{21}) & k(la_{22}) & k(la_{23}) \end{bmatrix}$$
$$=\begin{bmatrix} (kl)a_{11} & (kl)a_{12} & (kl)a_{13} \\ (kl)a_{21} & (kl)a_{22} & (kl)a_{23} \end{bmatrix}$$
$$=(kl)A$$

10. $$1A=\begin{bmatrix} 1a_{11} & 1a_{12} & 1a_{13} \\ 1a_{21} & 1a_{22} & 1a_{23} \end{bmatrix}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}=A$$

b) Let $\vec{u}=(u_1,u_2,u_3)$, $\vec{v}=(v_1,v_2,v_3)$ and $\vec{w}=(w_1,w_2,w_3)$

1. $\vec{u}+\vec{v}=(u_1+v_1,u_2+v_2,u_3+v_3)\in\mathbb{R}^3$
2. $\vec{u}+\vec{v}=(u_1+v_1,u_2+v_2,u_3+v_3)=(v_1+u_1,v_2+u_2,v_3+u_3)=\vec{v}+\vec{u}$
3. $$\vec{u}+(\vec{v}+\vec{w})=(u_1+(v_1+v_1),u_2+(v_2+w_2),u_3+(v_3+w_3))$$
$$=((u_1+v_1)+w_1,(u_2+v_2)+w_2,(u_3+v_3)+w_3)$$
$$=(\vec{u}+\vec{v})+\vec{w}$$
4. $\vec{u}+\vec{0}=(u_1,u_2,u_3)+(0,0,0)=(u_1+0,u_2+0,u_3+0)=(u_1,u_2,u_3)=\vec{u}$
5. $\vec{u}+(-\vec{u})=(u_1,u_2,u_3)+(-u_1,-u_2,-u_3)=(u_1-u_1,u_2-u_2,u_3-u_3)=(0,0,0)=\vec{0}$

6. $k\vec{u} = (ku_1, ku_2, ku_3) \in \mathbb{R}^3$
7. $k(\vec{u}+\vec{v}) = k(u_1+v_1, u_2+v_2, u_3+v_3)$
$= (k(u_1+v_1), k(u_2+v_2), k(u_3+v_3))$
$= (ku_1+kv_1, ku_2+kv_2, ku_3+kv_3)$
$= k\vec{u}+k\vec{v}$
8. $(k+l)\vec{u} = ((k+l)u_1, (k+l)u_2, (k+l)u_3)$
$= (ku_1+lu_1, ku_2+lu_2, ku_3+lu_3)$
$= k\vec{u}+l\vec{u}$
9. $k(l\vec{u}) = k(lu_1, lu_2, lu_3) = ((kl)u_1, (kl)u_2, (kl)u_3) = (kl)\vec{u}$
10. $1\vec{u} = (1u_1, 1u_2, 1u_3) = (u_1, u_2, u_3) = \vec{u}$

c) Let $\mathbf{p}(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ $\mathbf{q}(x) = b_3x^3 + b_2x^2 + b_1x + b_0$ and $\mathbf{r}(x) = c_3x^3 + c_2x^2 + c_1x + c_0$

1. $\mathbf{p}(x)+\mathbf{q}(x) = (a_3+b_3)x^3 + (a_2+b_2)x^2 + (a_1+b_1)x + (a_0+b_0)$ is a 3rd degree polynomial
2. $\mathbf{p}(x)+\mathbf{q}(x) = (a_3+b_3)x^3 + (a_2+b_2)x^2 + (a_1+b_1)x + (a_0+b_0)$
$= (b_3+a_3)x^3 + (b_2+a_2)x^2 + (b_1+a_1)x + (b_0+a_0)$
$= \mathbf{q}(x)+\mathbf{p}(x)$
3. $\mathbf{p}(x)+(\mathbf{q}(x)+\mathbf{r}(x)) = a_3x^3 + a_2x^2 + a_1x + a_0 + (b_3+c_3)x^3 + (b_2+c_2)x^2 + (b_1+c_1)x + (b_0+c_0)$
$= (a_3+(b_3+c_3))x^3 + (a_2+(b_2+c_2))x^2 + (a_1+(b_1+c_1))x + (a_0+(b_0+c_0))$
$= ((a_3+b_3)+c_3)x^3 + ((a_2+b_2)+c_2)x^2 + ((a_1+b_1)+c_1)x + ((a_0+b_0)+c_0)$
$= (a_3+b_3)x^3 + (a_2+b_2)x^2 + (a_1+b_1)x + (a_0+b_0) + c_3x^3 + c_2x^2 + c_1x + c_0$
$= (\mathbf{p}(x)+\mathbf{q}(x))+\mathbf{r}(x)$
4. $\mathbf{p}(x)+\mathbf{0} = (a_3+0)x^3 + (a_2+0)x^2 + (a_1+0)x + (a_0+0)$
$= a_3x^3 + a_2x^2 + a_1x + a_0$
$= \mathbf{p}(x)$
5. $\mathbf{p}(x) + (\text{-}\mathbf{p}(x)) = (a_3-a_3)x^3 + (a_2-a_2)x^2 + (a_1-a_1)x + (a_0+a_0)$
$= 0x^3 + 0x^2 + 0x + 0$
$= \mathbf{0}$
6. $k\mathbf{p}(x) = ka_3x^3 + ka_2x^2 + ka_1x + ka_0$ is a 3rd degree polynomial

7. $k\left(\mathbf{p}(x)+\mathbf{q}(x)\right)=k\left(\left(a_3+b_3\right)x^3+\left(a_2+b_2\right)x^2+\left(a_1+b_1\right)x+\left(a_0+b_0\right)\right)$
$=\left(ka_3+kb_3\right)x^3+\left(ka_2+kb_2\right)x^2+\left(ka_1+kb_1\right)x+\left(ka_0+kb_0\right)$
$=ka_3x^3+ka_2x^2+ka_1x+ka_0+kb_3x^3+kb_2x^2+kb_1x+kb_0$
$=k\mathbf{p}(x)+k\mathbf{q}(x)$
8. $\left(k+l\right)\mathbf{p}(x)=\left(k+l\right)a_3x^3+\left(k+l\right)a_2x^2+\left(k+l\right)a_1x+\left(k+l\right)a_0$
$=\left(ka_3x^3+ka_2x^2+ka_1x+ka_0\right)+\left(la_3x^3+la_2x^2+la_1x+la_0\right)$
$=k\mathbf{p}(x)+l\mathbf{p}(x)$
9. $k\left(l\mathbf{p}(x)\right)=k\left(la_3x^3+la_2x^2+la_1x+la_0\right)=kla_3x^3+kla_2x^2+kla_1x+kla_0=\left(kl\right)\mathbf{p}(x)$
10. $1\mathbf{p}(x)=1a_3x^3+1a_2x^2+1a_1x+1a_0=a_3x^3+a_2x^2+a_1x+a_0=\mathbf{p}(x)$

d) Axiom 5 is not satisfied, there is no $\mathbf{-}\vec{u}$ in the set such that $\vec{u}+\left(\mathbf{-}\vec{u}\right)=\vec{0}$ because $\mathbf{-}\vec{u}=\left(-1\right)\vec{u}=\left(-u_1,-u_2\right)$ is not in the set.

Axiom 6 is not satisfied because if $k<0$ then $k\vec{u}=\left(ku_1,ku_2\right)$ is not in the set, ku_1 and ku_2 being <0.

e) Axiom 1 is not satisfied since $A+B=\begin{bmatrix}a_{11} & 1\\ 1 & a_{22}\end{bmatrix}+\begin{bmatrix}b_{11} & 1\\ 1 & b_{22}\end{bmatrix}=\begin{bmatrix}a_{11}+b_{11} & 2\\ 2 & a_{22}+b_{22}\end{bmatrix}$ is not in the set.

Axiom 4 is not satisfied since $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ is not in the set.

Axiom 5 is not satisfied since $\mathbf{-}A=\begin{bmatrix}-a_{11} & -1\\ -1 & -a_{22}\end{bmatrix}$ is not in the set.

Axiom 6 in not satisfied since $kA=\begin{bmatrix}ka_{11} & k\\ k & ka_{22}\end{bmatrix}$ is not in the set if $k\neq 1$

f) Let $\mathbf{p}(x)=ax^5$, $\mathbf{q}(x)=bx^5$ and $\mathbf{r}(x)=cx^5$.

1. $\mathbf{p}(x)+\mathbf{q}(x)=ax^5+bx^5=\left(a+b\right)x^5$ is in the set
2. $\mathbf{p}(x)+\mathbf{q}(x)=\left(a+b\right)x^5=\left(b+a\right)x^5=\mathbf{q}(x)+\mathbf{p}(x)$
3. $\mathbf{p}(x)+\left(\mathbf{q}(x)+\mathbf{r}(x)\right)=ax^5+\left(b+c\right)x^5=\left(a+\left(b+c\right)\right)x^5=\left(\left(a+b\right)+c\right)x^5$
$=\left(a+b\right)x^5+cx^5=\left(\mathbf{p}(x)+\mathbf{q}(x)\right)+\mathbf{r}(x)$
4. $\mathbf{p}(x)+\mathbf{0}=\left(a+0\right)x^5=ax^5=\mathbf{p}(x)$
5. $\mathbf{p}(x)+\left(\mathbf{-p}(x)\right)=\left(a-a\right)x^5=0x^5=\mathbf{0}$
6. $k\mathbf{p}(x)=kax^5$ is in the set
7. $k\left(\mathbf{p}(x)+\mathbf{q}(x)\right)=k\left(\left(a+b\right)x^5\right)=\left(ka+kb\right)x^5=kax^5+kbx^5=k\mathbf{p}(x)+k\mathbf{q}(x)$

8. $(k+l)\mathbf{p}(x)=(k+l)ax^5=kax^5+lax^5=k\mathbf{p}(x)+l\mathbf{p}(x)$
9. $k(l\mathbf{p}(x))=k(lax^5)=klax^5=(kl)\mathbf{p}(x)$
10. $1\mathbf{p}(x)=1ax^5=ax^5=\mathbf{p}(x)$

g) The set is not a vector space since axiom 8 fails. For example, let $k=1,\ l=2$ and $\vec{u}=(1,1)$.

$$(k+l)\odot\vec{u}=(1+2)\odot(1,1)=((1+2)1,1)=(3,1)$$

$$(k\odot\vec{u})\oplus(l\odot\vec{u})=(1\odot(1,1))\oplus(2\odot(1,1))=(1,1)\oplus(2,1)=(3,2)$$

Thus $(k+l)\odot\vec{u}\neq(k\odot\vec{u})\oplus(l\odot\vec{u})$.

Axioms 4 and 5 also fail.

h) The set is not a vector space because axiom 2 fails. For example, let $\vec{u}=(1,2)$ and $\vec{v}=(2,1)$.

$$\vec{u}\oplus\vec{v}=(1,2)\oplus(2,1)=(1,0)$$

$$\vec{v}\oplus\vec{u}=(2,1)\oplus(1,2)=(2,0)$$

Thus $\vec{u}\oplus\vec{v}\neq\vec{v}\oplus\vec{u}$.

Axioms 4, 5 and 8 also fail.

i) Axiom 4 fails since $\vec{u}\oplus\vec{0}=(u_1\cdot 0,u_2\cdot 0)=(0,0)\neq\vec{u}$ if $\vec{u}\neq\vec{0}$

Axiom 5 and 7 also fail.

j) Axiom 8 fails

$$\begin{aligned}(k+l)\odot\vec{u}&=\left((k+l)^2u_1,(k+l)^2u_2,(k+l)^2u_3\right)\\&=\left(k^2u_1+2klu_1+l^2u_1,k^2u_2+2klu_2+l^2u_2,k^2u_3+2klu_3+l^2u_3\right)\\&=(k\odot\vec{u})\oplus\left(\sqrt{2kl}\odot\vec{u}\right)\oplus(l\odot\vec{u})\\&\neq(k\odot\vec{u})\oplus(l\odot\vec{u})\end{aligned}$$

Axiom 5 also fails.

4. Yes. It is similar to the vector space $V=\{\vec{0}\}$.

5. a) Yes 1. $\vec{u}+\vec{v}=(u_1,u_2,0)+(v_1,v_2,0)=(u_1+v_1,u_2+v_2,0)\in W$

2. $k\vec{u}=k(u_1,u_2,0)=(ku_1,ku_2,0)\in W$

Thus W is a subspace of $\mathbb{R}^3$

b) No 1. $\vec{u}+\vec{v}=(u_1,1,1)+(v_1,1,1)=(u_1+v_1,2,2)\notin W$

c) Yes 1. $\vec{u}+\vec{v}=(u_1,u_2,u_1+u_2)+(v_1,v_2,v_1+v_2)=(u_1+v_1,u_2+v_2,(u_1+v_1)+(u_2+v_2))\in W$

2. $k\vec{u}=k(u_1,u_2,u_1+u_2)=(ku_1,ku_2,ku_1+ku_2)\in W$

Thus W is a subspace of $\mathbb{R}^3$

d) No 2. $k\vec{u}=k(u_1,u_2,u_1u_2)=(ku_1,ku_2,ku_1u_2)\notin W$ since $(ku_1)(ku_2)=k^2u_1u_2\neq ku_1u_2$ if $k\neq 1$.

e) Yes 1. $\vec{u}+\vec{v}=(u_1,u_2-u_1,u_2)+(v_1,v_2-v_1,v_2)=(u_1+v_1,(u_2+v_2)-(u_1+v_1),u_2+v_2)\in W$

2. $k\vec{u}=k(u_1,u_2-u_1,u_2)=(ku_1,ku_2-ku_1,u_2)\in W$

Thus W is a subspace of $\mathbb{R}^3$

f) Yes If $\vec{u}\in W$ then $u_1-2u_2+u_3=0$ and if $\vec{v}\in W$ then $v_1-2v_2+v_3=0$.

1. $\vec{u}+\vec{v}=(u_1,u_2,u_3)+(v_1,v_2,v_3)=(u_1+v_1,u_2+v_2,u_3+v_v)$

Since $(u_1+v_1)-2(u_2+v_2)+(u_3+v_3)=(u_1-2u_2+u_3)+(v_1-2v_2+v_3)=0+0=0$

then $\vec{u}+\vec{v}\in W$.

2. $k\vec{u}=k(u_1,u_2,u_3)=(ku_1,ku_2,ku_3)$

Since $ku_1-2ku_2+ku_3=k(u_1-2u_2+u_3)=k0=0$

then $k\vec{u}\in W$.

Thus W is a subspace of $\mathbb{R}^3$

g) No If $\vec{u}\in W$ then $2u_1+u_2-u_3=3$ and if $\vec{v}\in W$ then $2v_1+v_2-v_3=3$.

1. $\vec{u}+\vec{v}=(u_1,u_2,u_3)+(v_1,v_2,v_3)=(u_1+v_1,u_2+v_2,u_3+v_v)$

Since $2(u_1+v_1)+(u_2+v_2)-(u_3+v_3)=(2u_1+u_2-u_3)+(2v_1+v_2-v_3)=3+3=6$

then $\vec{u}+\vec{v}\notin W$.

h) Yes If $\vec{u}\in W$ then $\vec{u}=(2t,-t,5t)$ and if $\vec{v}\in W$ then $\vec{v}=(2s,-s,5s)$.

1. $\vec{u}+\vec{v}=(2t,-t,5t)+(2s,-s,5s)$

$=(2t+2s,-t-s,5t+5s)=(2(t+s),-(t+s),5(t+s))\in W$

2. $k\vec{u}=k(2t,-t,5t)=(2kt,-kt,5kt)=(2(kt),-(kt),5(kt))\in W$

Thus W is a subspace of $\mathbb{R}^3$

i) Yes If $\vec{u},\vec{v}\in W$, then $\vec{u}\bullet\vec{w}=0$ and $\vec{v}\bullet\vec{w}=0$

1. $\vec{u}+\vec{v}\in W$ since $(\vec{u}+\vec{v})\bullet\vec{w}=\vec{u}\bullet\vec{w}+\vec{v}\bullet\vec{w}=0+0=0$

2. $k\vec{u}=\in W$ since $(k\vec{u})\bullet\vec{w}=k(\vec{u}\bullet\vec{w})=k0=0$

Thus W is a subspace of $\mathbb{R}^3$

6. a) No. We do not always have closure under scalar multiplication.

For a example if $k=\frac{1}{2}$ and A=$\begin{bmatrix}1&1\\1&1\end{bmatrix}\in W$, then $\frac{1}{2}\begin{bmatrix}1&1\\1&1\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{bmatrix}\notin W$

b) No. Let A=$\begin{bmatrix}1&0\\0&0\end{bmatrix}$ and $B=\begin{bmatrix}0&0\\0&1\end{bmatrix}$. Then $\det(A)=\det(B)=0$ so $A,B\in W$.

Since $\det(A+B)=\begin{vmatrix}1&0\\0&1\end{vmatrix}=1$, then $A+B\notin W$, so we do not have closure under addition.

c) Yes. Let A and B be symmetric matrices, $A^T=A$ and $B^T=B$.

1. $(A+B)^T=A^T+B^T=A+B$, hence $A+B\in W$

2. $(kA)^T = kA^T = kA$, hence $kA \in W$

Hence W is a subspace of $M_{2,2}$.

d) Yes. Let $A=\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$ and $B=\begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \end{bmatrix}$ be in W.

1. $A+B=\begin{bmatrix} a_{11}+b_{11} & 0 \\ 0 & a_{22}+b_{22} \end{bmatrix} \in W$

2. $kA=\begin{bmatrix} ka_{11} & 0 \\ 0 & ka_{22} \end{bmatrix} \in W$

Hence W is a subspace of $M_{2,2}$.

e) Yes 1. $A+B=\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}+\begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix}=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ 0 & a_{22}+b_{22} \end{bmatrix} \in W$

2. $kA=k\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}=\begin{bmatrix} ka_{11} & ka_{12} \\ 0 & ka_{22} \end{bmatrix} \in W$

Hence W is a subspace of $M_{2,2}$.

f) Yes 1. $A+B=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}+\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} \in W$ since

$$a_{11}+b_{11}+a_{12}+b_{12}+a_{21}+b_{21}+a_{22}+b_{22}=\left(a_{11}+a_{12}+a_{21}+a_{22}\right)+\left(b_{11}+b_{12}+b_{21}+b_{22}\right)=0$$

2. $kA=k\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}=\begin{bmatrix} ka_{11} & ka_{12} \\ ka_{22} & ka_{22} \end{bmatrix} \in W$ since

$$ka_{11}+ka_{12}+ka_{21}+ka_{22}=k\left(a_{11}+a_{12}+a_{21}+a_{22}\right)=0$$

Hence W is a subspace of $M_{2,2}$.

7. a) No. We do not always have closure under scalar multiplication.
For a example if $k=-1$, then $-f(x) \notin W$ since $-f(x)<0$

b) Yes. 1. $(\mathbf{f}+\mathbf{g})(x) \in W$ since $(\mathbf{f}+\mathbf{g})(-x)=\mathbf{f}(-x)+\mathbf{g}(-x)=\mathbf{f}(x)+\mathbf{g}(x)=(\mathbf{f}+\mathbf{g})(x)$

2. $(k\mathbf{f})(x) \in W$ since $(k\mathbf{f})(-x)=k\mathbf{f}(-x)=k\mathbf{f}(x)=(k\mathbf{f})(x)$

c) Yes. 1. $(\mathbf{f}+\mathbf{g})(x) \in W$ since $(\mathbf{f}+\mathbf{g})(-x)=\mathbf{f}(-x)+\mathbf{g}(-x)=-\mathbf{f}(x)-\mathbf{g}(x)=-(\mathbf{f}+\mathbf{g})(x)$

2. $(k\mathbf{f})(x) \in W$ since $(k\mathbf{f})(-x)=k\mathbf{f}(-x)=-k\mathbf{f}(x)=-(k\mathbf{f})(x)$

d) Yes. 1. $(\mathbf{f}+\mathbf{g})(x) \in W$ since $(\mathbf{f}+\mathbf{g})(x)=\mathbf{f}(x)+\mathbf{g}(x)=c+d$ is a constant.

2. $(k\mathbf{f})(x) \in W$ since $(k\mathbf{f})(x)=k\mathbf{f}(x)=kc$ is a constant.

e) Yes. 1. $(\mathbf{f}+\mathbf{g})(x) \in W$ since $(\mathbf{f}+\mathbf{g})(0)=\mathbf{f}(0)+\mathbf{g}(0)=0+0=0$.

2. $(k\mathbf{f})(x) \in W$ since $(k\mathbf{f})(0)=k\mathbf{f}(0)=k0=0$.

f) No . 1. $(\mathbf{f}+\mathbf{g})(x) \notin W$ since $(\mathbf{f}+\mathbf{g})(0)=\mathbf{f}(0)+\mathbf{g}(0)=1+1=2$.

2. $\left(k\mathbf{f}\right)(x) \notin W$ since $\left(k\mathbf{f}\right)(0) = k1 = k \neq 1$ if $k \neq 1$.