MATHEMATICS 201-NYC-05
Vectors and Matrices
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Fall 2007

XIII - Planes in $\mathbb{R}^3$

1. Find, if possible, an equation for the plane in
   i) vector form
   ii) point-normal form
   iii) general form

   a) Passing through P(5,-1,2) and parallel to the vectors $\vec{u}=(2,7,-4)$ and $\vec{v}=(2,-1,5)$.
   b) Passing through the points P(-2,1,5), Q(-2,1,-3) and R(1,1,4).
   c) Passing through the points P(1,0,6), Q(-3,4,7) and R(2,0,12).
   d) Passing through the point P(2,-5,5) and having $\vec{n}=(1,-2,1)$.
   e) Passing through the point P(3,-1,8) and containing the line $L:(x,y,z)=(-1,2,5)+t(1,1,-2) \quad t\in\mathbb{R}$.
   f) Containing the lines $L_1:\begin{cases} x=6+4t \\ y=2-t \\ z=4+3t \end{cases} \quad t\in\mathbb{R}$ and $L_2:\dfrac{x-2}{3}=\dfrac{3-y}{2}=z-1$.
   g) Containing the lines $L_1:(x,y,z)=(-1,2,4)+t(4,-1,3),\ t\in\mathbb{R}$ and $L_2:\dfrac{x-2}{3}=\dfrac{3-y}{2}=z-1$.
   h) Containing the lines $L_1:(-1,2,2)+t(2,1,-3) \quad t\in\mathbb{R}$ and $L_2:\dfrac{x-2}{2}=y-1=\dfrac{2-z}{3}$.
   i) Passing through the point P(3,-2,5) and perpendicular to the line $L:(x,y,z)=(-1,2,2)+t(3,2,1) \quad t\in\mathbb{R}$.
   j) Containing the line $L_1:\dfrac{x+3}{2}=y+5=2z$ and perpendicular to the plane $\pi:2x-4y+z-1=0$.
   k) Passing through P(2,-1,4) and parallel to the plane $\pi:3x-4y+z-5=0$
   l) Passing through P(2,-3,5) and parallel to the *xz*-plane.
   m) Passing through P(-1,1,5) and containing the intersection of the planes $\pi_1:15x+y+9z=62$ and $\pi_2:\begin{cases} x=6+5s+3t \\ y=-10+8s+t \\ z=-2+3s+t \end{cases} \quad s,t\in\mathbb{R}$.
   n) Parallel to the *yz*-plane and passing through the point P of intersection between the line $L:(x,y,z)=(-1,3,3)+t(3,1,5),\ t\in\mathbb{R}$ and the plane $\pi:2x-4y+z-10=0$.

2. Consider the following planes.

$$\pi_1 : x - 2y - z - 4 = 0$$

$$\pi_2 : (x, y, z) = (0, 2, -1) + r(-1, 3, 3) + t(1, 1, 0) \qquad s, t \in \mathbb{R}$$

$$\pi_3 : \begin{cases} x = -2 + 4s - 2t \\ y = s - 3t \\ z = 5 + 2s + 4t \end{cases} \qquad s, t \in \mathbb{R}$$

$$\pi_4 : 2(x-1) + 3(y-5) - 4(z+1) = 0$$

a) Find the angle between the planes π_1 and π_2, the planes π_1 and π_3 and the planes π_1 and π_4.
b) Find which planes are parallel or perpendicular with π_1.
c) Find, if possible, the intersection of the planes π_1 and π_2, the planes π_1 and π_3 and the planes π_1 and π_4.
d) Find the relative position of each pair of planes.
e) Determine the distance from each plane to the point *P*(1,2,-3).
f) Determine the distance between the pair of planes π_1 and π_3.
g) For each of the points *A*(1,7,2), *B*(15,9,-7) and *C*(1,1,1), determine to which plane (if any) they belong.
h) Find the point on each plane that is closest to the point *P*(1,4,2).
i) For each plane, find the equation of the line passing through the point *P*(1,2,-3) and perpendicular to the plane.

3. Prove that the distance from the origin to the plane $\pi : ax + by + cz + d = 0$ is given by

$$d(0, \pi) = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$$

4. Prove that the distance from $P_0(x_0, y_0, z_0)$ to the plane $\pi : ax + by + cz + d = 0$ is given by

$$d(P_0, \pi) = \frac{|ax_0 + by_0 + cz_0 d|}{\sqrt{a^2 + b^2 + c^2}}$$

5. Prove that the distance between the parallel planes $\pi_1 : ax + by + cz + d_1 = 0$ and $\pi_2 : ax + by + cz + d_2 = 0$ is given by

$$d(\pi_1, \pi_2) = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

6. Consider the line $L : \frac{x-1}{3} = \frac{2-y}{2} = z + 1$

a) Find the equation of the plane perpendicular to *L* passing through $P(2, -1, 3)$.
b) Find the intersection between *L* and the plane found in (a).

7. Consider the plane $\pi : 2x-3y+z-3=0$.
    a) Find the equation of the line passing through the point $P(3,-1,5)$ and perpendicular to the plane π.
    b) Find the point Q on the plane π that is closest to the point $P(3,-1,5)$.
    c) Find the distance between P and the plane π.
    d) Find the equation for the line passing through the point $P(3,-1,5)$ and parallel to the planes π and $\pi_2 : x-y+3z-5=0$.
    e) Find the equation of the plane parallel to π and passing through the point $P(3,-1,5)$.

Answers

1. a) (i) $(x,y,z)=(5,-1,2)+s(2,7,-4)+t(2,-1,5) \quad s,t\in\mathbb{R}$
    (ii) $31(x-5)-18(y+1)-16(z-2)=0$ (iii) $31x-18y-16z-141=0$
    b) (i) $(x,y,z)=(-2,1,5)+s(0,0,-8)+t(3,0,-1) \quad s,t\in\mathbb{R}$
    (ii) $-24(y-1)=0$ (iii) $y-1=0$
    c) (i) $(x,y,z)=(1,0,6)+s(-4,4,1)+t(1,0,6) \quad s,t\in\mathbb{R}$
    (ii) $24(x-1)+25y-4(z-6)=0$ (iii) $24x+25y-4z=0$
    d) (ii) $(x-2)-2(y+5)+(z-5)=0$ (iii) $x-2y+z-17=0$
    (i) $(x,y,z)=(2,-5,5)+s(2,1,0)+t(-1,0,1) \quad s,t\in\mathbb{R}$
    e) (i) $(x,y,z)=(3,-1,8)+s(1,1,-2)+t(-4,3,-3) \quad s,t\in\mathbb{R}$
    (ii) $3(x-3)+11(y+1)+7(z-8)=0$ (iii) $3x+11y+8z-54=0$
    f) L_1 and L_2 intersect at P(2,3,1) (i) $(x,y,z)=(2,3,1)+s(4,-1,3)t(3,-2,1) \quad s,t\in\mathbb{R}$
    (ii) $5(x+1)+5(y-2)-5(z-5)$ (iii) $x+y-z-4=0$
    g) L_1 and L_2 do not intersect, thus there are no planes containing the lines L_1 and L_2.
    h) (i) $(x,y,z)=(-1,2,2)+s(3,-1,0)+t(2,1,-3) \quad s,t\in\mathbb{R}$
    (ii) $3(x+1)+9(y-2)+5(z-2)=0$ (iii) $3x+9y+5z-25=0$
    i) (ii) $3(x-3)+2(y+2)+(z-5)=0$ iii) $3x+2y+z-10=0$
    (iii) $(x,y,z)=(3,-2,5)+s(-2,3,0)+t(-1,0,3) \quad s,t\in\mathbb{R}$
    j) (i) $(x,y,z)=(-3,-5,0)+s(2,1,\frac{1}{2})+t(2,-4,1) \quad s,t\in\mathbb{R}$
    (ii) $3(x+3)-(y+5)-10z=0$ (iii) $3x-y-10z+4=0$
    k) (ii) $3(x-2)-4(y+1)+(z-4)=0$ (iii) $3x-4y+z-14=0$

(i) $(x,y,z)=(2,-1,4)+s(4,3,0)+t(-1,0,3) \quad s,t\in\mathbb{R}$

l) (i) $(x,y,z)=(2,-1,4)+s(1,0,0)+t(0,0,1) \quad s,t\in\mathbb{R}$

(ii) $0(x-2)-(y+1)+0(z-4)=0$ (iii) $y=-1$

m) $\pi_1\cap\pi_2:(x,y,z)=(4,2,0)+t(-1,6,1)$, thus taking the points (4,2,0) and (3,8,1) from $\pi_1\cap\pi_2$ we have (i) $(x,y,z)=(-1,1,5)+s(5,1,-5)+t(4,7,-4) \quad s,t\in\mathbb{R}$

(ii) $31(x+1)+31(z-5)=0$ (iii) $x+z-4=0$

l) $\pi\cap L=\{(8,6,18)\}$ (i) $(x,y,z)=(8,6,18)+s(0,1,0)+t(0,0,1) \quad s,t\in\mathbb{R}$

(ii) $1(x-8)+0(y-6)+0(z-18)=0$ (iii) $x=8$

2. a) 69.5° 0° 90°

b) $\pi_1 // \pi_3$ and $\pi_1\perp\pi_4$.

c) $\pi_1\cap\pi_2:(x,y,z)=\left(-\frac{32}{3},-\frac{22}{3},0\right)+t\left(-\frac{11}{3},-\frac{7}{3},1\right) \quad t\in\mathbb{R}$ $\pi_1\cap\pi_3:\varnothing$

$\pi_1\cap\pi_4:(x,y,z)=\left(\frac{54}{7},\frac{13}{7},0\right)+t\left(\frac{11}{7},\frac{2}{7},1\right) \quad t\in\mathbb{R}$

d) $\pi_1 // \pi_3$ and distinct, all other pairs are intersecting, with

$\pi_1\cap\pi_2:(x,y,z)=\left(-\frac{32}{3},-\frac{22}{3},0\right)+t(11,7,-3)$, $\pi_1\cap\pi_4:(x,y,z)=\left(\frac{54}{7},\frac{13}{7},0\right)+t(11,2,7)$,

$\pi_2\cap\pi_3:(x,y,z)=\left(\frac{1}{3},\frac{11}{3},0\right)+t(11,7-3)$, $\pi_2\cap\pi_4:(x,y,z)=\left(\frac{11}{5},\frac{83}{15},0\right)+t(0,4,3)$,

$\pi_3\cap\pi_4:(x,y,z)=(3,5,0)+t(11,2,7)$

e) $\frac{2\sqrt{6}}{3}$ $\frac{5\sqrt{34}}{34}$ $\frac{7\sqrt{6}}{6}$ $\frac{\sqrt{29}}{29}$ f) $\frac{11\sqrt{6}}{6}$

g) $A\in\pi_2$ $B\in\pi_1$ and $B\in\pi_2$ C is not on any of the planes

h) $\left(\frac{19}{6},\frac{-1}{3},\frac{-1}{6}\right)$ $\left(\frac{7}{34},\frac{163}{34},\frac{16}{17}\right)$ $\left(\frac{4}{3},\frac{10}{3},\frac{5}{3}\right)$ $\left(\frac{59}{29},\frac{161}{29},\frac{-2}{29}\right)$

i) $x-1=\frac{2-y}{2}=-z-3$ $\frac{1-x}{3}=\frac{y-2}{3}=\frac{-z-3}{4}$ $x-1=\frac{2-y}{2}-z-3$ $\frac{x-1}{2}=\frac{y-2}{3}=\frac{-z-3}{4}$

3. We have $\vec{n}=(a,b,c)$. Let us take $R\left(-\frac{d}{a},0,0\right)$.

$$d(0,\pi)=\frac{\left|\overrightarrow{RP}\bullet\vec{n}\right|}{\|\vec{n}\|}=\frac{\left|\left(-\frac{d}{a}\right)\bullet(a,b,c)\right|}{\sqrt{a^2+b^2+c^2}}=\frac{|d|}{\sqrt{a^2+b^2+c^2}}$$

4. We have $\vec{n}=(a,b,c)$. Let us take $R\left(-\frac{d}{a},0,0\right)$.

$$d(P_0,\pi)=\frac{\left|\overrightarrow{RP_0}\bullet\vec{n}\right|}{\|\vec{n}\|}=\frac{\left|\left(x_0+\frac{d}{a},y_0,z_0\right)\bullet(a,b,c)\right|}{\sqrt{a^2+b^2+c^2}}=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

5. We have $\vec{n}=(a,b,c)$. Let us take $P_1\left(-\frac{d_1}{a},0,0\right)$ and $P_2\left(-\frac{d_2}{a},0,0\right)$.

$$d(\pi_1,\pi_2)=\frac{\left|\overrightarrow{P_1P_2}\bullet\vec{n}\right|}{\|\vec{n}\|}=\frac{\left|\left(-\frac{d_2}{a}+\frac{d_1}{a},0,0\right)\bullet(a,b,c)\right|}{\sqrt{a^2+b^2+c^2}}=\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$$

6. a) $3x-2y+z-11=0$ b) $\left(\frac{53}{14},\frac{1}{7},\frac{-1}{14}\right)$

7. a) $\dfrac{x-3}{2}=\dfrac{-y-1}{3}=z-5$ b) $\left(\frac{10}{7},\frac{19}{14},\frac{59}{14}\right)$ c) $\frac{11\sqrt{14}}{14}$

d) $\dfrac{3-x}{8}=\dfrac{-y-1}{5}=z-5$ e) $2x-3y+z-14=0$