

## MATHEMATICS 201-NYC-05

Vectors and Matrices

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# XIII - Cross Product

- Let  $\vec{u} = (1, -2, 3)$ ,  $\vec{v} = (4, 5 - 1)$  and  $\vec{w} = (-2, 1, 7)$ . Compute
  - $\vec{u} \times \vec{v}$
  - $\vec{v} \times \vec{w}$
  - $\vec{u} \times (\vec{v} \times \vec{w})$
  - $(\vec{u} \times \vec{v}) \times \vec{w}$
  - $(\vec{u} \times \vec{v}) \times (\vec{v} \times \vec{w})$
  - $\vec{u} \times (\vec{v} - 2\vec{w})$
- Find a vector that is orthogonal to both  $\vec{u}$  and  $\vec{v}$ .
  - $\vec{u} = (1, -2, 3)$ ,  $\vec{v} = (4, 5 - 1)$
  - $\vec{u} = (4, -3, 5)$ ,  $\vec{v} = (-8, 0, 6)$
- Extend  $S$  to form a basis for  $\mathbb{R}^3$ .
  - $S = \{(-2, 1, 5), (3, 1, 1)\}$
  - $S = \{(1, 1, 2), (2, 1, 1)\}$
- Find the area of the parallelogram determined by  $\vec{u}$  and  $\vec{v}$ .
  - $\vec{u} = (3, 0)$ ,  $\vec{v} = (1, 1)$
  - $\vec{u} = (\frac{1}{2}, -\frac{2}{3})$ ,  $\vec{v} = (4, 3)$
  - $\vec{u} = (4, 0, -3)$ ,  $\vec{v} = (0, -2, 5)$
  - $\vec{u} = (1, -3, 2)$ ,  $\vec{v} = (-1, -1, -1)$
- Find the area of the triangle with vertices
  - $A(1, 0)$ ,  $B(2, 3)$  and  $C(6, 0)$
  - $A(6, 5)$ ,  $B(1, 3)$  and  $C(3, -2)$
  - $A(1, 2, 3)$ ,  $B(-1, -3, 2)$  and  $C(5, 0, 5)$
- Find the scalar triple product  $\vec{u} \cdot (\vec{v} \times \vec{w})$ .
  - $\vec{u} = (1, -2, 3)$ ,  $\vec{v} = (4, 5 - 1)$ ,  $\vec{w} = (-1, 2, 0)$
  - $\vec{u} = (2, 2, 2)$ ,  $\vec{v} = (0, -2, 1)$ ,  $\vec{w} = (2, -2, 4)$
- Let  $\vec{u} \cdot (\vec{v} \times \vec{w}) = 2$ . Find
  - $\vec{u} \cdot (\vec{w} \times \vec{v})$
  - $(\vec{v} \times \vec{w}) \cdot \vec{u}$
  - $\vec{w} \cdot (\vec{u} \times \vec{v})$
  - $\vec{v} \cdot (\vec{u} \times \vec{w})$
  - $(\vec{u} \times \vec{w}) \cdot \vec{v}$
  - $\vec{u} \cdot (\vec{w} \times \vec{w})$
- Find the volume of the parallelepiped with sides  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$ .
  - $\vec{u} = (1, -2, 3)$ ,  $\vec{v} = (4, 5 - 1)$ ,  $\vec{w} = (-2, 1, 7)$ .
  - $\vec{u} = (1, -3, 2)$ ,  $\vec{v} = (-1, -1, -1)$ ,  $\vec{w} = (2, -2, 4)$

9. Find the volume of the tetrahedron with vertices A, B, C and D.
- $A(-1,4,4), B(-1,-3,2), C(5,0,5), D(2,-3,4)$
  - $A(3,-1,3), B(2,2,-1), C(1,-1,3), D(2,3,4)$
10. Determine whether  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  lie in the same plane when positioned so that their initial points coincide.
- $\vec{u} = (2,6,6), \vec{v} = (4,5-1), \vec{w} = (-2,1,7)$
  - $\vec{u} = (1,-3,2), \vec{v} = (-1,-1,-1), \vec{w} = (5,-1,-2)$
11. Prove that if  $\theta$  is the angle between  $\vec{u}$  and  $\vec{v}$ , where  $\vec{u}$  and  $\vec{v}$  are nonorthogonal vectors in  $\mathbf{R}^3$ , then
- $$\tan \theta = \frac{\|\vec{u} \times \vec{v}\|}{\vec{u} \bullet \vec{v}}$$
12. Prove the following identities for vectors  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  in  $\mathbf{R}^3$ .
- $(\vec{u} + k\vec{v}) \times \vec{v} = \vec{u} \times \vec{v}$
  - $(\vec{u} + \vec{v}) \times (\vec{u} - \vec{v}) = -2(\vec{u} \times \vec{v})$

## Answers

- $(-13,13,13)$
  - $(36,-26,14)$
  - $(50,94,46)$
  - $(78,65,13)$
  - $(520,650,-130)$
  - $(21,39,19)$
- $(-13,13,13)$
  - $(-18,-64,-24)$
- $S' = \{(-2,1,5), (3,1,1), (-4,17,-5)\}$
  - $S' = \{(1,1,2), (2,1,1), (-1,3,-1)\}$
- 3
  - $\frac{25}{6}$
  - $10\sqrt{5}$
  - $\sqrt{42}$
- $\frac{15}{2}$
  - $\frac{29}{2}$
  - $6\sqrt{5}$
- 39
  - 0
- 2
  - 2
  - 2
  - 2
  - 2
  - 0
- 130
  - 4
- $\frac{13}{2}$
  - $\frac{19}{3}$
- Yes
  - No

11. Since  $\theta \neq 90^\circ$ , then  $\|\vec{u}\|\|\vec{v}\| \cos \theta = \vec{u} \bullet \vec{v}$ . Also, since  $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin \theta$  then

$$\|\vec{u} \times \vec{v}\| = \frac{\vec{u} \bullet \vec{v}}{\cos \theta} \sin \theta \text{ which gives us } \tan \theta = \frac{\|\vec{u} \times \vec{v}\|}{\vec{u} \bullet \vec{v}}.$$

- $(\vec{u} + k\vec{v}) \times \vec{v} = \vec{u} \times \vec{v} + (k\vec{v} \times \vec{v}) = \vec{u} \times \vec{v} + k(\vec{v} \times \vec{v}) = \vec{u} \times \vec{v} + k\mathbf{0} = \vec{u} \times \vec{v}$
  - $(\vec{u} + \vec{v}) \times (\vec{u} - \vec{v}) = (\vec{u} + \vec{v}) \times \vec{u} - (\vec{u} + \vec{v}) \times \vec{v} = \vec{u} \times \vec{u} + \vec{v} \times \vec{u} - \vec{u} \times \vec{v} - \vec{v} \times \vec{v} = 0 - \vec{u} \times \vec{v} - \vec{u} \times \vec{v} - 0 = -2(\vec{u} \times \vec{v})$