MATHEMATICS 201-NYC-05
Vectors and Matrices
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XII - Lines in $\mathbb{R}^3$

1. Find, if possible, an equation for the line in
   i) vector form
   ii) parametric form
   iii) symmetric form

   a) Passing through P(5,-1,2) and parallel to the vector $\vec{u} = (2,7,-4)$.
   b) Passing through the points P(-2,3,5) and Q(4,1,-3).
   c) Passing through the points P(1,0,6) and Q(-3,0,7).
   d) Passing through the points P(3,-1,8) and Q(-6,2,-16).
   e) Passing through the point P(1,5,1) and parallel to $\begin{cases} x = -1+4t \\ y = 2-t \\ z = 5+3t \end{cases}, \quad t \in \mathbb{R}$.
   f) Passing through the point P(2,4,1) and parallel to $\dfrac{x+5}{2} = \dfrac{2-y}{3} = 4z$.
   g) Passing through the point P(3,-2,6) and parallel to $(x,y,z) = (-2,1,4) + t(-1,2,2) \ : \ t \in \mathbb{R}$.
   h) Passing through P(-1,3,5) and perpendicular to the lines $L_1 : \dfrac{x+3}{2} = y+5 = 2z$ and $L_2 : (x,y,z) = (-1,2,2) + t(1,2,-5), \quad t \in \mathbb{R}$
   i) Passing through P(2,-1,4) and perpendicular to the lines $L_3 : \dfrac{x}{2} = \dfrac{y-3}{2} = \dfrac{z+1}{3}$ and $L_4 : \begin{cases} x = -3+t \\ y = 2-5t, \\ z = 4+t \end{cases} \quad t \in \mathbb{R}$.
   j) Passing through P(2,-3,5) and perpendicular to the *xz*-plane.
   k) Passing through P(-1,1,5) and the intersection of the lines $L_1 : (x,y,z) = (1,2,3) + t(-2,7,-1), \quad t \in \mathbb{R}$ and $L_2 : \begin{cases} x = 4+t \\ y = 5+10t, \\ z = 6+2t \end{cases} \quad t \in \mathbb{R}$.

2. Consider the following lines.

$$L_1 : \frac{x-1}{2} = \frac{y}{-2} = -2z$$

$$L_2 : (x, y, z) = (0, 2, -1) + t(1,1,0), \quad t \in \mathbb{R}$$

$$L_3 : \begin{cases} x = -1 + 2t \\ y = 3 \\ z = t \end{cases}, \quad t \in \mathbb{R}$$

$$L_4 : \frac{x-3}{2} = y - 3 = \frac{2-z}{3}$$

$$L_5 : (x, y, z) = (3, -2, 1) + t(4, -4, -1), \quad t \in \mathbb{R}$$

a) Find the angle between the line L_1 and L_2, L_1 and L_3, L_1 and L_4, L_1 and L_5.
b) Find which lines are parallel or perpendicular with L_1.
c) Find, if possible, the intersection of the lines L_2 and L_4, L_3 and L_4.
d) Find the relationship between each pair of lines.
e) Determine the distance from each line to the point $P(1,2,-3)$.
f) Find the point Q on each line that is closest to the point $P(-1,2,2)$.
g) Determine the distance between the pair of lines L_1 and L_5, and L_2 and L_3.
h) For each of the points $A(5,3,3)$, $B(1,1,1)$ and $C(-1,2,2)$, determine to which line they belong.

3. Consider the lines $L_1 : \frac{x-d}{3} = \frac{2y-4}{3} = \frac{16z+20}{b}$ and $L_2 : \frac{3x+6}{a} = \frac{4-y}{2} = \frac{4z+8}{3}$.
a) Find the value of the constants a and b for which the lines are parallel.
b) Find the value of d for which the lines are equivalent.

4. Find the point P on the line $\frac{x+1}{-1} = \frac{y}{2} = \frac{z-2}{4}$ that is closest to the origin.

5. Consider the line $L : 5 - x = y + 4 = \frac{z+3}{2}$.
a) Find the point Q on the line L that is closest to the point $P(8,9,1)$.
b) Find the distance between the point P and the line L.
c) Find the equation of a line perpendicular to L and passing through the point P.

Answers

1. a) $(x,y,z)=(5,-1,2)+t(2,7,-4),\quad t\in\mathbb{R}$

$$\begin{cases} x=5+2t \\ y=-1+7t, \quad t\in\mathbb{R} \\ z=2-4t \end{cases}$$

$$\frac{x-5}{2}=\frac{y+1}{7}=\frac{2-z}{4}$$

b) $(x,y,z)=(-2,3,5)+t(6,-2,-8),\quad t\in\mathbb{R}$

$$\begin{cases} x=-2+6t \\ y=3-2t\ , \quad t\in\mathbb{R} \\ z=5-8t \end{cases}$$

$$\frac{x+2}{6}=\frac{3-y}{2}=\frac{5-z}{8}$$

c) $(x,y,z)=(1,0,6)+t(-4,0,1),\quad t\in\mathbb{R}$

$$\begin{cases} x=1-4t \\ y=0\ , \quad t\in\mathbb{R} \\ z=6+t \end{cases}$$

None, but we do have $\frac{1-x}{4}=z-6,\quad y=0$

d) $(x,y,z)=t(3,-1,8),\quad t\in\mathbb{R}$

$$\begin{cases} x=3t \\ y=-t, \quad t\in\mathbb{R} \\ z=8t \end{cases}$$

$$\frac{x}{3}=\frac{y}{-1}=\frac{z}{8}$$

e) $(x,y,z)=(1,5,1)+t(4,-1,3),\quad t\in\mathbb{R}$

$$\begin{cases} x=1+4t \\ y=5-t\ , \quad t\in\mathbb{R} \\ z=1+3t \end{cases}$$

$$\frac{x-1}{4}=5-y=\frac{z-1}{3}$$

f) $(x,y,z)=(2,4,1)+t(2,-3,\tfrac{1}{4}),\quad t\in\mathbb{R}$

$$\begin{cases} x=2+2t \\ y=4-3t, \quad t\in\mathbb{R} \\ z=1+\tfrac{1}{4}t \end{cases}$$

$$\frac{x-2}{2}=\frac{4-y}{3}=4z-4$$

g) $(x,y,z)=(3,-2,6)+t(-1,2,2),\quad t\in\mathbb{R}$

$$\begin{cases} x=3-t \\ y=-2+2t, \quad t\in\mathbb{R} \\ z=6+2t \end{cases}$$

$$3-x=\frac{y+2}{2}=\frac{z-6}{2}$$

h) $(x,y,z)=(-1,3,5)+t\left(-6,\tfrac{21}{2},3\right),\quad t\in\mathbb{R}$ $\quad\begin{cases}x=-1-6t\\ y=3+\tfrac{21}{2}t\\ z=5+3t\end{cases},\quad t\in\mathbb{R}$

$$\frac{x+1}{-6}=\frac{2y-6}{21}=\frac{z-5}{3}$$

i) $(x,y,z)=(2,-1,4)+t(17,1,-12),\quad t\in\mathbb{R}$ $\quad\begin{cases}x=2+17t\\ y=-1+t\\ z=4-12t\end{cases},\quad t\in\mathbb{R}$

$$\frac{x-2}{17}=y+1=\frac{4-z}{12}$$

j) $(x,y,z)=(2,-3,5)+t(0,1,0),\quad t\in\mathbb{R}$ $\quad\begin{cases}x=2\\ y=-3+t\\ z=5\end{cases},\quad t\in\mathbb{R}$

None, but we have $x = 2,\ z = 5$

k) $(x,y,z)=(-1,1,5)+t(4,-6,-1),\quad t\in\mathbb{R}$ $\quad\begin{cases}x=-1+4t\\ y=1-6t\\ z=5-t\end{cases},\quad t\in\mathbb{R}$

$$\frac{x+1}{4}=\frac{1-y}{6}=5-z$$

2. a) $90°,\ 57°,\ 71°,\ 0°$
   b) $L_1 \perp L_2 \quad L_1 // L_5$
   c) none, $(3,3,2)$
   d) L_1 and L_5 are parallel and distinct, L_3 and L_4 are concurrent with intersection (3,3,2) and all other pair of lines are skew.
   e) $\frac{2\sqrt{3333}}{33},\ \frac{3\sqrt{2}}{2},\ \frac{\sqrt{345}}{5},\ \frac{4\sqrt{70}}{14},\ \frac{2\sqrt{6501}}{33}$
   f) $Q_1\left(\frac{-13}{11},\frac{24}{11},\frac{6}{11}\right),\ Q_2\left(\frac{-1}{2},\frac{3}{2},-1\right),\ Q_3\left(\frac{-1}{5},3,\frac{2}{5}\right),\ Q_4\left(\frac{12}{7},\frac{33}{14},\frac{55}{14}\right),\ Q_5(-1,2,2)$
   g) $\frac{2\sqrt{66}}{11},\ \frac{2\sqrt{6}}{3}$
   h) $A\in L_3$ and $C\in L_5$
3. a) $a = -12,\ b = -9$ b) $d = -6$
4. $\left(\frac{-4}{7},\frac{-6}{7},\frac{2}{7}\right)$
5. a) $P(2,-1,3)$ b) $2\sqrt{35}$ c) $\frac{x-8}{6}=\frac{y-9}{10}=z+2$