MATHEMATICS 201-NYC-05
Vectors and Matrices
Martin Huard
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XI - Lines in $\mathbb{R}^2$

1. Find, if possible, an equation for the line in
   i) vector form
   ii) parametric form
   iii) symmetric form
   iv) general form
   v) slope-intercept form ($y = mx + b$)
   a) Passing through P(-1,2) and parallel to the vector $\vec{u} = (-2,3)$.
   b) Passing through the points P(2,5) and Q(1,-3).
   c) Passing through the points P(4,-1) and Q(2,2).
   d) Passing through the points P(2,-3) and Q(4,-3).
   e) Passing through P(3,-4) and perpendicular to the vector $\vec{n} = (1,-5)$.
   f) Passing through the point P(1,1) and parallel to $y = 3x - 5$.
   g) Passing through the point P(2,4) and parallel to $\frac{x-1}{2} = \frac{3-y}{5}$.
   h) Passing through the point P(3,-2) and parallel to $2x - 5y + 4 = 0$.
   i) Passing through P and Q, where P is the intersection of the lines $L_1 : \frac{x+3}{2} = y + 5$ and $L_2 : (x, y) = (2,2) + t(1,2),\ \ t \in \mathbb{R}$ and Q is the intersection of the lines $L_3 : 5x - 2y - 3 = 0$ and $L_4 : \begin{cases} x = 7 + 6t \\ y = -4 - 5t \end{cases} \quad t \in \mathbb{R}$.
   j) Passing through P(2,0) and tangent to the circle $C : x^2 + y^2 = 2$.

2. Consider the following lines.

$$L_1 : \frac{x+3}{2} = y + 5$$

$$L_2 : (x, y) = (2,2) + t(-2,4),\ \ t \in \mathbb{R}$$

$$L_3 : 2x + y - 6 = 0$$

$$L_4 : \begin{cases} x = 7 + t \\ y = -4 \end{cases} \quad t \in \mathbb{R}$$

   a) Find the angle between the lines
   i) L_1 and L_2 ii) L_1 and L_3 iii) L_1 and L_4 iv) L_2 and L_3.
   b) Find, if possible, the intersection of the lines
   i) L_1 and L_2 ii) L_1 and L_3 iii) L_1 and L_4
   c) Find the relative position of each pair for each possible pair of lines.

d) Determine the distance from each line to the point *P*(2,-3).
e) Find the point *Q* that is closest to the point *P*(-2,1).
f) Determine the distance between each pair of parallel lines found in (b).
g) For each of the points *A*(-2,1), *B*(1,4) and *C*(-3,-4), determine to which line they belong.

3. Consider the lines $L_1 : ax + y - 3 = 0$ and $L_2 : 9x + ay - 9 = 0$ Find the value of *a* such that the lines are
   a) The same b) Parallel and distinct
   c) Perpendicular c) Neither parallel nor perpendicular.

4. Prove that the distance between the origin and the line given by the equation $ax + by + c = 0$ is $\dfrac{|c|}{\sqrt{a^2 + b^2}}$.

5. Prove that the distance between the point $P(x_0, y_0)$ and the line given by the equation $ax + by + c = 0$ is $\dfrac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$.

6. Prove that the distance between the parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is $\dfrac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$.

7. Find the equation of all lines parallel to the line $L : 3x - 4y + 7 = 0$ with a distance of 3 units from it.

Answers

1. a) $(x,y)=(-1,2)+t(-2,3),\ t\in\mathbb{R}$, $\begin{cases}x=-1-2t\\ y=2+3t\end{cases}$ $t\in\mathbb{R}$, $\frac{x+1}{-2}=\frac{y-2}{3}$,
   $3x+2y-1=0$, $y=-\frac{3}{2}x+\frac{1}{2}$

   b) $(x,y)=(2,5)+t(-1,-8),\ t\in\mathbb{R}$, $\begin{cases}x=2-t\\ y=5-8t\end{cases}$ $t\in\mathbb{R}$, $2-x=\frac{5-y}{8}$,
   $8x-y-11=0$, $y=8x-11$

   c) $(x,y)=(4,-1)+t(-2,3),\ t\in\mathbb{R}$, $\begin{cases}x=4-2t\\ y=-1+3t\end{cases}$ $t\in\mathbb{R}$, $\frac{x-4}{-2}=\frac{y+1}{3}$,
   $3x+2y-10=0$, $y=-\frac{3}{2}x+5$

   d) $(x,y)=(2,-3)+t(2,0),\ t\in\mathbb{R}$, $\begin{cases}x=2+2t\\ y=-3\end{cases}$ $t\in\mathbb{R}$, no symmetric form,
   $y+3=0$, $y=-3$

   e) $(x,y)=(3,-4)+t(5,1),\ t\in\mathbb{R}$, $\begin{cases}x=3+5t\\ y=-4+t\end{cases}$ $t\in\mathbb{R}$, $\frac{x-3}{5}=y+4$,
   $x-5y-23=0$, $y=\frac{1}{5}x-\frac{23}{5}$

   f) $(x,y)=(1,1)+t(1,3),\ t\in\mathbb{R}$, $\begin{cases}x=1+t\\ y=1+3t\end{cases}$ $t\in\mathbb{R}$, $x-1=\frac{y-1}{3}$,
   $3x-y-2=0$, $y=3x-2$

   g) $(x,y)=(2,4)+t(2,-5),\ t\in\mathbb{R}$, $\begin{cases}x=2+2t\\ y=4-5t\end{cases}$ $t\in\mathbb{R}$, $\frac{x-2}{2}=\frac{4-y}{5}$,
   $5x+2y-18=0$, $y=-\frac{5}{2}x+9$

   h) $(x,y)=(3,-2)+t(5,2),\ t\in\mathbb{R}$, $\begin{cases}x=3+5t\\ y=-2+2t\end{cases}$ $t\in\mathbb{R}$, $\frac{x-3}{5}=\frac{y+2}{2}$,
   $2x-5y-16=0$, $y=\frac{2}{5}x-\frac{16}{5}$

   i) $(x,y)=(-1,-4)+t(2,5),\ t\in\mathbb{R}$, $\begin{cases}x=-1+2t\\ y=-4+5t\end{cases}$ $t\in\mathbb{R}$, $\frac{x+1}{2}=\frac{y+4}{5}$,
   $5x-2y-3=0$, $y=\frac{5}{2}x-\frac{3}{2}$

   j) $(x,y)=(2,0)+t(-1,1),\ t\in\mathbb{R}$, $\begin{cases}x=2-t\\ y=t\end{cases}$ $t\in\mathbb{R}$, $2-x=y$,
   $x+y-2=0$, $y=-x+2$

2. a) $90°$, $90°$, $26.6°$, $0°$
   b) $\left(\frac{19}{5},\frac{-8}{5}\right)$, $\left(\frac{19}{5},\frac{-8}{5}\right)$, $(-1,-4)$
   c) $L_2 \,//\, L_3$ and identical $\quad L_1\perp L_2 \quad L_1\perp L_3 \quad L_1$, L_2 and L_3 are concurrent to L_4

d) $d(P,L_1)=\frac{\sqrt{5}}{5}$ $\quad d(P,L_2)=\sqrt{5}$ $\quad d(P,L_3)=\sqrt{5}$ $\quad d(P,L_4)=1$

e) $Q_1\left(\frac{1}{5},\frac{-17}{5}\right)$ $\quad Q_2\left(\frac{8}{5},\frac{14}{5}\right)$ $\quad Q_3\left(\frac{8}{5},\frac{14}{5}\right)$ $\quad Q_4(-2,-4)$

f) 0

g) A: none B: L_2 and L_3 C: L_4

3. a) 3 b) –3 c) 0 d) $\mathbb{R}\setminus\{-3,0,3\}$

4. We have $P(0,0)$ and $\vec{n}=(a,b)$. If we take $R\left(0,\frac{-c}{b}\right)$, then $\overrightarrow{PR}=\left(0,\frac{-c}{b}\right)$. Thus

$$d=\frac{\left|\overrightarrow{PR}\bullet\vec{n}\right|}{\|\vec{n}\|}=\frac{\left|\left(0,\frac{-c}{b}\right)\bullet(a,b)\right|}{\|(a,b)\|}=\frac{|c|}{\sqrt{a^2+b^2}}$$

5. We have $\vec{n}=(a,b)$. If we take $R\left(0,\frac{-c}{b}\right)$, then $\overrightarrow{PR}=\left(-x_0,-y_0-\frac{c}{b}\right)$. Thus

$$d=\frac{\left|\overrightarrow{PR}\bullet\vec{n}\right|}{\|\vec{n}\|}=\frac{\left|\left(-x_0,-y_0-\frac{c}{b}\right)\bullet(a,b)\right|}{\|(a,b)\|}=\frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$$

6. We have $\vec{n}=(a,b)$. If we take $P_1\left(0,-\frac{c_1}{b}\right)$ and $P_2\left(0,\frac{-c_2}{b}\right)$, then $\overrightarrow{P_1P_2}=\left(0,\frac{-c_2+c_1}{b}\right)$. Thus

$$d=\frac{\left|\overrightarrow{PR}\bullet\vec{n}\right|}{\|\vec{n}\|}=\frac{\left|\left(0,\frac{-c_2+c_1}{b}\right)\bullet(a,b)\right|}{\|(a,b)\|}=\frac{|c_1-c_2|}{\sqrt{a^2+b^2}}$$

7. $3x-4y-8=0$ or $3x-4y+22=0$