

MATHEMATICS 201-NYC-05

Vectors and Matrices

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V - Adjoins and Cramer's Rule

1. Find the adjoint of the matrix A . Then use the adjoint to find the inverse of A , if possible.

a) $A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

b) $A = \begin{bmatrix} -2 & 5 \\ 3 & -10 \end{bmatrix}$

c) $A = \begin{bmatrix} 2 & -1 & 4 \\ 6 & 3 & -2 \\ 0 & 3 & 1 \end{bmatrix}$

d) $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 5 & 2 \\ 0 & -10 & -4 \end{bmatrix}$

e) $A = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix}$

f) $A = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 3 & -1 & 4 & 0 \\ 0 & 0 & 1 & 2 \\ -1 & 1 & 0 & 2 \end{bmatrix}$

2. Use Cramer's Rule to solve the given system of linear equations, if possible.

a) $3x - 2y = 2$
 $x + 4y = 7$

b) $4x - y - z = 1$
 $2x + 2y + 3z = 10$
 $5x - 2y - 2z = -1$

c) $3x + 3y + 5z = 1$
 $3x + 5y + 9z = 2$
 $5x + 9y + 17z = 4$

3. Use Cramer's Rule to solve for x_2 .

a) $2x_1 - 3x_2 + x_3 = 3$
 $x_1 + x_2 - x_3 = -1$
 $3x_3 - 2x_2 + 5x_3 = 8$

b) $x_1 - x_2 + 3x_3 - x_4 = 2$
 $5x_2 + 2x_4 = 1$
 $2x_1 - 2x_3 - x_4 = 5$
 $x_1 + x_2 + 4x_3 = 9$

4. Use the adjoint to find the inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if $ad - bc \neq 0$.

5. Find the inverse of $A = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$ using the adjoint. (The determinant of this matrix was calculated in the previous exercise sheet.)

6. Prove that if A is an invertible matrix of order $n > 1$ then $\det(\text{Adj} A) = (\det A)^{n-1}$.

7. Find the inverse of $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ using the adjoint (assuming $adf \neq 0$). From your result,

what can we say about the inverse of a triangular matrix with nonzero entries in the main diagonal?

Answers

1. a) $\text{Adj}(A) = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} \frac{-1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix}$ b) $\text{Adj}(A) = \begin{bmatrix} -10 & -5 \\ -3 & -2 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} -2 & -1 \\ \frac{-3}{5} & \frac{-2}{5} \end{bmatrix}$

c) $\text{Adj}(A) = \begin{bmatrix} 9 & 13 & -10 \\ -6 & 2 & 28 \\ 18 & -6 & 12 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} \frac{3}{32} & \frac{13}{96} & \frac{-5}{48} \\ \frac{-1}{16} & \frac{1}{48} & \frac{7}{24} \\ \frac{3}{16} & \frac{-1}{16} & \frac{1}{8} \end{bmatrix}$ d) $\text{Adj}(A) = \begin{bmatrix} 0 & -26 & -13 \\ 0 & -8 & -4 \\ 0 & 20 & 10 \end{bmatrix}$, A^{-1} does not exist

e) $\text{Adj}(A) = \begin{bmatrix} 3 & 1 & -17 \\ 0 & 2 & -10 \\ 0 & 0 & 6 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} & \frac{-17}{6} \\ 0 & \frac{1}{3} & \frac{-5}{3} \\ 0 & 0 & 1 \end{bmatrix}$

d) $\text{Adj}(A) = \begin{bmatrix} 6 & -3 & 12 & -15 \\ 2 & 1 & -4 & 3 \\ -4 & 4 & -10 & 12 \\ 2 & -2 & 8 & -6 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1 & \frac{-1}{2} & 2 & \frac{-5}{2} \\ \frac{1}{3} & \frac{1}{6} & \frac{-2}{3} & \frac{1}{2} \\ \frac{-2}{3} & \frac{2}{3} & \frac{-5}{3} & 2 \\ \frac{1}{3} & \frac{-1}{3} & \frac{4}{3} & -1 \end{bmatrix}$

2. a) $x = \frac{11}{7}$, $y = \frac{19}{14}$ b) $x = 1$, $y = 1$, $z = 2$ c) $x = 0$, $y = -\frac{1}{2}$, $z = \frac{1}{2}$

3. a) $x_2 = -\frac{7}{25}$ b) $x_2 = -\frac{35}{3}$

4. $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ 5. $A^{-1} = \begin{bmatrix} \frac{bc}{(b-a)(c-a)} & \frac{-ac}{(b-a)(c-b)} & \frac{ab}{(c-a)(c-b)} \\ \frac{-(b+c)}{(b-a)(c-a)} & \frac{a+c}{(b-a)(c-b)} & \frac{-(a+b)}{(c-a)(c-b)} \\ \frac{1}{(b-a)(c-a)} & \frac{-1}{(b-a)(c-b)} & \frac{1}{(c-a)(c-b)} \end{bmatrix}$

6. We have $A \cdot \text{Adj}(A) = \det(A)I$. Thus $\det(A \cdot \text{Adj}(A)) = \det(\det(A)I)$

$$\det(A) \det(\text{Adj}(A)) = (\det(A))^n \det(I)$$

$$\det(\text{Adj}(A)) = (\det(A))^{n-1}$$

7. $A^{-1} = \begin{bmatrix} \frac{1}{a} & \frac{-b}{ad} & \frac{be-cd}{adf} \\ 0 & \frac{1}{d} & \frac{-e}{df} \\ 0 & 0 & \frac{1}{f} \end{bmatrix}$ Hence, the inverse of an upper triangular matrix is also an upper

triangular matrix, if the inverse exists.