MATHEMATICS 201-NYC-05
Vectors and Matrices
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Thursday November 29, 2005

TEST #4
SOLUTIONS

Question 1 (5 points)

Show that the set $V = \{(x, y) : x, y \in \mathbb{R}\}$ with the following operations is not a vector space.

$$(u_1, u_2) \oplus (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$$
$$k \odot (u_1, u_2) = (ku_1, 2u_2)$$

Axiom 10 fails.

Counter Example: Let $\vec{u} = (3,4)$. Then $1 \odot \vec{u} = 1 \odot (3,4) = (3,8) \neq \vec{u}$

(Axioms 4, 5, 7 and 9 also fail)

Question 2 (8 points)

Determine if W is a subspace of V.

a) $W = \{(a, 2a+b, b); a, b \in \mathbb{R}\}$ $V = \mathbb{R}^3$

W is nonempty since $(0,0,0) \in W$

Let $\vec{u} = (a_1, 2a_1 + b_1, b_1) \in W$ and $\vec{v} = (a_2, 2a_2 + b_2, b_2) \in W$

1. $\vec{u} + \vec{v} = (a_1, 2a_1 + b_1, b_1) + (a_2, 2a_2 + b_2, b_2)$
$= (a_1 + a_2, 2a_1 + b_1 + 2a_2 + b_2, b_1 + b_2)$
$= (a_1 + a_2, 2(a_1 + a_2) + (b_1 + b_2), b_1 + b_2) \in W$

2. $k\vec{u} = k(a_1, 2a_1 + b_1, b_1)$
$= (ka_1, k(2a_1 + b_1), kb_1)$
$= (ka_1, 2(ka_1) + kb_1, kb_1) \in W$

Hence W is a subspace of $\mathbb{R}^3$

b) $W = \{A : A^2 = I, A \in M_{2,2}\}$ $V = M_{2,2}$?

No. If $A = I$ and $B = I$, then $A, B \in W$ since $I^2 = I$ but

$(A+B)^2 = (I+I)^2 = (2I)^2 = 4I \neq I$ thus $A + B \notin W$

Question 3 (7 points)

Consider the set $S=\{x^2+3, 2x^2+x+6, 2x+1\}$.

a) Does S span P_2? Support your answer.

Let $p(x)=ax^2+bx+c\in P_2$

$c_1(x^2+3)+c_2(2x^2+x+6)+c_3(2x+1)=ax^2+bx+c$

$$\left[\begin{array}{ccc|c}1&2&0&a\\0&1&2&b\\3&6&1&c\end{array}\right]\xrightarrow{R_3\to R_3-2R_1}\left[\begin{array}{ccc|c}1&2&0&a\\0&1&2&b\\0&0&1&c-3a\end{array}\right]$$

Thus $c_3=c-3a$

$c_2=6a+b-2c$

$c_1=-11a-2b+4c$

Hence S spans P_2.

b) Is S a basis for P_2? Support your answer.

Yes, since $n(S)=3=\dim(P_2)$ and S spans P_2 then S is a basis for P_2.

Question 4 (9 points)

Are the following sets S bases for the vector space V? Support your answer.

a) $S=\{x^4-x^2, x^3+x\}$, $V=P_5$

No since $n(S)=2\neq\dim(P_5)=6$

b) $S=\{(1,3,2),(0,1,1),(4,2,-2)\}$, $V=\mathbb{R}^3$

$c_1(1,3,2)+c_2(0,1,1)+c_3(4,2,-2)=(0,0,0)$

$$\left[\begin{array}{ccc|c}1&0&4&0\\3&1&2&0\\2&1&-2&0\end{array}\right]\xrightarrow[R_3\to R_3-2R_1]{R_2\to R_2-3R_1}\left[\begin{array}{ccc|c}1&0&4&0\\0&1&-10&0\\0&1&-10&0\end{array}\right]\xrightarrow{R_3\to R_3-R_2}\left[\begin{array}{ccc|c}1&0&4&0\\0&1&-10&0\\0&0&0&0\end{array}\right]$$

Thus $c_3=t$, $c_2=10t$, $c_1=-4t$, hence S is linearly dependent.

Ergo S is not a basis for V.

c) $S=\{(2,3),(-1,5),(4,4),(-8,1)\}$, $V=\mathbb{R}^2$

No since $n(S)=4\neq\dim(\mathbb{R}^2)=2$

Question 5 (9 points)

Let $W=\left\{\begin{bmatrix} a & a+b \\ 0 & a+2b \end{bmatrix} : a,b\in\mathbb{R}\right\}$

a) Find a basis for W.

$$\begin{bmatrix} a+b & a+b \\ 0 & a+2b \end{bmatrix} = a\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

If $S=\left\{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\right\}$ then $W=\operatorname{span}(S)$.

Since S is linearly independent (the two matrices are not multiples of each other) then S is a basis for W.

b) What is the dimension of W?

$\dim(W)=2$

c) Find the coordinate vector of $A=\begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix}$ relative to the basis found in (a).

$$c_1\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{array}\right] \xrightarrow[R_3\to R_3-R_1]{R_2\to R_2-R_1} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{array}\right] \xrightarrow{R_3\to R_3-2R_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

Thus $c_2=-2$ and $c_1=3$.

Hence $(A)_S=(3,-2)$

Question 6 (12 points)

Let $A=\{(1,1,2),(2,3,4),(0,1,0)\}$ and $B=\{(3,1,5)\}$. Find a basis (if possible) and the dimension for each of the following vector spaces, and give a geometrical interpretation (planes in general form and lines in symmetric form).

a) $\operatorname{span}(A)$

$c_1(1,1,2)+c_2(2,3,4)+c_3(0,1,0)=(0,0,0)$

$$\left[\begin{array}{ccc|c}1&2&0&0\\1&3&1&0\\2&4&0&0\end{array}\right]\xrightarrow[R_3\to R_3-2R_1]{R_2\to R_2-R_1}\left[\begin{array}{ccc|c}1&2&0&0\\0&1&1&0\\0&0&0&0\end{array}\right]\qquad \begin{array}{l}c_3=t\\c_2=-t\\c_1=2t\end{array}$$

If $t=1$, then $(0,1,0)=-2(1,1,2)+(2,3,4)$.

By the +/- theorem, if $S_A=\{(1,1,2),(2,3,4)\}$, then $\operatorname{span}(A)=\operatorname{span}(S_A)$.

Since S_A is linearly independent (the two vectors are not multiples of each other), then S_A is a basis for $\operatorname{span}(A)$ and $\dim(\operatorname{span}(A))=2$

Geometrically, $\vec{n}=\begin{vmatrix}i&j&k\\1&1&2\\2&3&4\end{vmatrix}=(-2,0,1)$, $\operatorname{span}(A)$ is the plane $2x-z=0$

b) $\operatorname{span}(B)$

B is a basis for $\operatorname{span}(B)$ since it is linearly independent.

Geometrically, $\operatorname{span}(B)$ is the line $\frac{x}{3}=y=\frac{z}{5}$ and $\dim(\operatorname{span}(B))=1$.

c) $\operatorname{span}(A)\cap\operatorname{span}(B)$

$(x,y,z)\in\operatorname{span}(A)\Rightarrow(x,y,z)=c_1(1,1,2)+c_2(2,3,4)$

$(x,y,z)\in\operatorname{span}(B)\Rightarrow(x,y,z)=c_3(3,1,5)$

$c_1(1,1,2)+c_2(2,3,4)+c_3(-3,-1,-5)=(0,0,0)$

$$\left[\begin{array}{ccc|c}1&2&-3&0\\1&3&-1&0\\2&4&-5&0\end{array}\right]\xrightarrow[R_3\to R_3-2R_1]{R_2\to R_2-R_1}\left[\begin{array}{ccc|c}1&2&-3&0\\0&1&2&0\\0&0&1&0\end{array}\right]\qquad \begin{array}{l}c_3=0\\c_2=0\\c_1=0\end{array}$$

Hence $(x,y,z)=(0,0,0)$

Ergo, $\operatorname{span}(A)\cap\operatorname{span}(B)=\{\vec{0}\}$ and $\dim(\operatorname{span}(A)\cap\operatorname{span}(B))=0$. There is no basis. Geometrically, $\operatorname{span}(A)\cap\operatorname{span}(B)$ is the origin.

d) Based on your answers in (a), (b) and (c), what is $\dim(\operatorname{span}(A)+\operatorname{span}(B))$? What is the geometrical interpretation? No work is expected. Just an answer!

$\dim(\operatorname{span}(A)+\operatorname{span}(B))=3$

$\operatorname{span}(A)+\operatorname{span}(B)=\mathbb{R}^3$