

MATHEMATICS 201-NYC-05

Vectors and Matrices

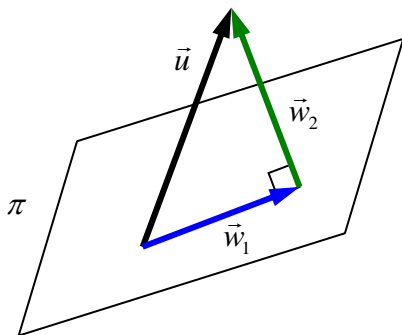
Martin Huard

November 7, 2005

Test #3 SOLUTIONS

Question 1 (6 points)

Consider the plane $\pi : 2x - 2y + z = 0$. Find \vec{w}_1 , the **orthogonal projection** of $\vec{u} = (6, -4, 7)$ on the plane π , as well as \vec{w}_2 the **component of \vec{u} orthogonal to π** .



$$\vec{n} = (2, -2, 1)$$

$$\vec{w}_2 = \text{proj}_{\vec{n}} \vec{u} = \frac{\vec{u} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n} = \frac{(6, -4, 7) \cdot (2, -2, 1)}{(2, -2, 1) \cdot (2, -2, 1)} (2, -2, 1) = \frac{27}{9} (2, -2, 1) = (6, -6, 3)$$

$$\vec{w}_1 = \vec{u} - \vec{w}_2 = (6, -4, 7) - (6, -6, 3) = (0, 2, 4)$$

Question 2 (6 points)

Consider the points $A(2, -1, 3)$, $B(1, 1, 2)$, $C(1, 0, 4)$ and $D(0, -1, 2)$.

- a) Find the volume of the tetrahedron with vertices A , B , C and D .

$$\overline{AB} = (-1, 2, -1)$$

$$\overline{AC} = (-1, 1, 1)$$

$$\overline{AD} = (-2, 0, -1)$$

$$V = \frac{1}{6} |\overline{AB} \cdot (\overline{AC} \times \overline{AD})| = \frac{1}{6} \left| \begin{vmatrix} -1 & 2 & -1 \\ -1 & 1 & 1 \\ -2 & 0 & -1 \end{vmatrix} \right| = \frac{1}{6} \left| -2 \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + 0 - \begin{vmatrix} -1 & 2 \\ -1 & 1 \end{vmatrix} \right| = \frac{1}{6} |-7| = \frac{7}{6}$$

- b) Find the equation of the plane π (**in general form**) containing the points A , B and C .

$$\vec{n} = \overline{AB} \times \overline{AC} = \begin{vmatrix} i & j & k \\ -1 & 2 & -1 \\ -1 & 1 & 1 \end{vmatrix} = (3, 2, 1) \qquad \pi : 3x + 2y + z = 7$$

Question 3 (17 points)

Consider the plane $\pi : 2x - 4y + 3z = 1$ and the point $P(-3, 6, -9)$.

- a) Find the equation of the line l (*in symmetric form*) perpendicular to π and passing through the point P .

$$\vec{u} = \vec{n} = (2, -4, 3)$$

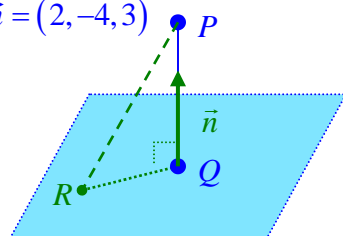
$$l: \frac{x+3}{2} = \frac{6-y}{4} = \frac{z+9}{3}$$

- b) Find the point Q on the plane π closest to the point P .

$$R(0, -1, -1)$$

$$\overrightarrow{PR} = (3, -7, 8)$$

$$\vec{n} = (2, -4, 3)$$



$$\overrightarrow{PQ} = \text{proj}_{\vec{n}} \overrightarrow{PR} = \frac{\vec{n} \cdot \overrightarrow{PR}}{\vec{n} \cdot \vec{n}} \vec{n}$$

$$(x+3, y-6, z+9) = \frac{58}{29} (2, -4, 3)$$

$$(x, y, z) = 2(2, -4, 3) + (-3, 6, -9)$$

$$(x, y, z) = (1, -2, -3)$$

$$\text{Hence } Q(1, -2, -3)$$

- c) Find the distance between the point P and the plane π .

$$d = \frac{|\overrightarrow{PR} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{58}{\sqrt{29}} = 2\sqrt{29}$$

- d) Find the equation of the plane π in *vector form*.

$$z = t$$

$$y = s$$

$$x = \frac{1}{2} + 2s - \frac{3}{2}t$$

$$\pi : (x, y, z) = \left(\frac{1}{2}, 0, 0\right) + s(2, 1, 0) + t\left(\frac{-3}{2}, 0, 1\right)$$

- e) Find the equation of the plane π_2 (*in general form*) parallel to the plane π and passing through the point P .

$$2x - 4y + 3z = 2(-3) - 4(6) + 3(-9)$$

$$\pi_2 : 2x - 4y + 3z = -57$$

Question 4 (13 points)

Consider the lines $L_1: \frac{x+12}{2} = y+3 = 1-z$ and $L_2: \begin{cases} x=2 \\ y=9-5t \\ z=-13+7t \end{cases}$.

- a) Find the relationship between lines L_1 and L_2 . That is, are they parallel and distinct, identical, intersecting or skew? If they are intersecting, find the point of intersection.

$$\begin{aligned} \vec{u}_1 &= (2, 1, -1) \\ \vec{u}_2 &= (0, 5, -7) \end{aligned} \quad \therefore \vec{u}_1 \not\parallel \vec{u}_2 \quad \therefore l_1 \not\parallel l_2$$

$$L_1: (x, y, z) = (-12, -3, 1) + t(2, 1, -1)$$

$$L_2: (x, y, z) = (2, 9, -13) + s(0, -5, 7)$$

$$(-12, -3, 1) + t(2, 1, -1) = (2, 9, -13) + s(0, -5, 7)$$

$$t(2, 1, -1) + s(0, 5, -7) = (14, 12, -14)$$

$$\left[\begin{array}{cc|c} 2 & 0 & 14 \\ 1 & 5 & 12 \\ -1 & -7 & -14 \end{array} \right] \begin{array}{l} R_2 \rightarrow 2R_2 - R_1 \\ R_3 \rightarrow 2R_3 + R_1 \end{array} \left[\begin{array}{cc|c} 2 & 0 & 14 \\ 0 & 10 & 10 \\ 0 & -14 & -14 \end{array} \right]$$

$$\begin{array}{l} R_3 \rightarrow 5R_3 + 7R_2 \\ R_1 \rightarrow \frac{1}{2}R_1 \\ R_2 \rightarrow \frac{-1}{10}R_2 \end{array} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array} \right]$$

$$t = 7, \quad s = 1$$

$$(x, y, z) = (2, 9, -13) + (0, -5, 7) = (2, 4, -6)$$

$$\text{Thus } L_1 \cap L_2 = \{(2, 4, -6)\}$$

- b) Find the point Q on the line L_1 that is closest to the point $P(2, 7, -9)$.

$$\vec{u}_1 = (2, 1, -1)$$

$$R(-12, -3, 1)$$

$$\vec{RP} = (14, 10, -10)$$

$$\vec{RQ} = \text{proj}_{\vec{u}} \vec{RP}$$

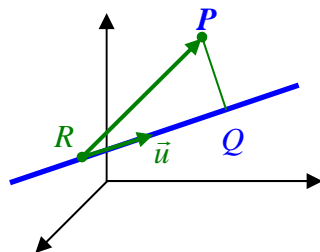
$$= \frac{\vec{RP} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u}$$

$$(x+12, y+3, z-1) = \frac{48}{6}(2, 1, -1)$$

$$(x, y, z) = 8(2, 1, -1) + (-12, -3, 1)$$

$$(x, y, z) = (4, 5, -7)$$

$$\text{Hence } Q(4, 5, -7)$$



- c) Find the angle θ between L_1 and L_2 .

$$\cos \theta = \frac{|\vec{u}_1 \cdot \vec{u}_2|}{\|\vec{u}_1\| \|\vec{u}_2\|} = \frac{|-12|}{\sqrt{6}\sqrt{74}}$$

$$\theta \approx 55.3^\circ$$

Question 5 (8 points)

Consider the lines $L_1: \frac{x-3}{2} = \frac{2-y}{3} = z+1$ and $L_2: \frac{1-x}{4} = \frac{y+1}{6} = \frac{3-z}{2}$.

- a) Find the equation of the plane π (**in general form**), if any, containing the lines L_1 and L_2 .

$$\vec{u}_1 = (2, -3, 1)$$

$$\vec{u}_2 = (-4, 6, -2)$$

$$\therefore \vec{u}_1 = -\frac{1}{2}\vec{u}_2 \quad \therefore l_1 \parallel l_2$$

$$P_1(3, 2, -1) \in L_1 \quad \frac{1-3}{4} \neq \frac{2+1}{6} \neq \frac{3-(-1)}{2}, \text{ thus } P_1 \notin L_2$$

Hence L_1 and L_2 are parallel and distinct.

$$P_2(1, -1, 3) \quad \overline{P_1P_2} = (-2, -3, 4)$$

$$\vec{n} = \overline{P_1P_2} \times \vec{u} = \begin{vmatrix} i & j & k \\ -2 & -3 & 4 \\ 2 & -3 & 1 \end{vmatrix} = (9, 10, 12)$$

$$9x + 10y + 12z = 9(3) + 10(2) + 12(-1)$$

$$\pi: 9x + 10y + 12z = 35$$

- b) Find the distance between L_1 and L_2 .

$$d = \frac{\|\overline{P_1P_2} \times \vec{u}\|}{\|\vec{u}\|} = \frac{\sqrt{325}}{\sqrt{11}} = \frac{5\sqrt{143}}{11}$$