

MATHEMATICS 201-NYC-05

Vectors and Matrices

Martin Huard

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Review Exercises
SOLUTIONS

1. Consider the matrices $A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 5 & -1 \\ 0 & -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 & 2 \\ -3 & 4 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & -1 \\ 1 & 6 \end{bmatrix}$. Evaluate, if

possible. (For f, g, and h, use your answer from (e)).

$$\text{a) } BA = \begin{bmatrix} -1 & 2 & 2 \\ -3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 \\ 1 & 5 & -1 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 8 & -3 \\ -2 & 24 & -12 \end{bmatrix}$$

$$\begin{aligned} \text{b) } B^T B - 3A &= \begin{bmatrix} -1 & -3 \\ 2 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 \\ -3 & 4 & 1 \end{bmatrix} - 3 \begin{bmatrix} 2 & -2 & 3 \\ 1 & 5 & -1 \\ 0 & -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 10 & -14 & -5 \\ -14 & 20 & 8 \\ -5 & 8 & 5 \end{bmatrix} - \begin{bmatrix} 6 & -6 & 9 \\ 3 & 15 & -3 \\ 0 & -6 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -8 & -14 \\ -17 & 5 & 11 \\ -5 & 14 & 2 \end{bmatrix} \end{aligned}$$

$$\text{c) } CB = \begin{bmatrix} 2 & -1 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 \\ -3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ -19 & 26 & 8 \end{bmatrix}$$

$$\begin{aligned} \text{d) } \text{tr}(3BB^T + C) &= \text{tr} \left(3 \begin{bmatrix} -1 & 2 & 2 \\ -3 & 4 & 1 \end{bmatrix} \begin{bmatrix} -1 & -3 \\ 2 & 4 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 1 & 6 \end{bmatrix} \right) \\ &= \text{tr} \left(3 \begin{bmatrix} 9 & 13 \\ 13 & 26 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 1 & 6 \end{bmatrix} \right) \\ &= \text{tr} \left(\begin{bmatrix} 29 & 38 \\ 40 & 84 \end{bmatrix} \right) \\ &= 29 + 84 = 113 \end{aligned}$$

$$\text{e) } \det(A) = \begin{vmatrix} 2 & -2 & 3 \\ 1 & 5 & -1 \\ 0 & -2 & 1 \end{vmatrix} = 2 \begin{vmatrix} 5 & -1 \\ -2 & 1 \end{vmatrix} - \begin{vmatrix} -2 & 3 \\ -2 & 1 \end{vmatrix} + 0 \begin{vmatrix} -2 & 3 \\ 5 & -1 \end{vmatrix} = 2 \cdot 3 - (4) + 0 = 2$$

$$\text{f) } \det(A^{17}) = [\det(A)]^{17} = 2^{17} = 131072$$

$$\text{g) } \det(5A^{-6}) = 5^3 \frac{1}{\det(A)^6} = \frac{125}{2^6} = \frac{125}{64}$$

$$\text{h) } \det(AA^T) = \det(A)\det(A^T) = \det(A)\det(A) = 2^2 = 4$$

$$\text{i) } \det(\text{adj}(A)) = \det([\text{cof}(A)]^T) = \det(\text{cof}(A))$$

$$\begin{aligned} &= \begin{vmatrix} 3 & -1 & -2 \\ -4 & 2 & 4 \\ -13 & 5 & 12 \end{vmatrix} \\ &= 3 \begin{vmatrix} 2 & 4 \\ 5 & 12 \end{vmatrix} + \begin{vmatrix} -4 & 4 \\ -13 & 12 \end{vmatrix} - 2 \begin{vmatrix} -4 & 2 \\ -13 & 5 \end{vmatrix} \\ &= 3 \cdot 4 + 4 - 2 \cdot 6 = 4 \end{aligned}$$

2. A square matrix A is called **skew-symmetric** if $A^T = -A$.

a) Prove that if A is invertible and skew-symmetric, then A^{-1} is skew-symmetric.

$$\text{To prove: } (A^{-1})^T = -A^{-1}$$

$$\begin{aligned} LS &= (A^{-1})^T \\ &= (A^T)^{-1} \\ &= (-A)^{-1} \quad \text{since } A \text{ is skew-symmetric } (A^T = -A) \\ &= -A^{-1} \\ &= RS \end{aligned}$$

b) Prove that A^T , $A+B$ and kA are skew-symmetric if A and B are skew symmetric.

$$\text{To prove: } (A^T)^T = -A^T$$

$$\begin{aligned} LS &= (A^T)^T \\ &= (-A)^T \quad \text{since } A \text{ is skew-symmetric} \\ &= -A^T \\ &= RS \end{aligned}$$

$$\text{To prove: } (A+B)^T = -(A+B)$$

$$\begin{aligned} LS &= (A+B)^T \\ &= A^T + B^T \\ &= -A - B \quad \text{since } A \text{ and } B \text{ are skew-symmetric} \\ &= -(A+B) \\ &= RS \end{aligned}$$

To prove: $(kA)^T = -(kA)$

$$\begin{aligned} LS &= (kA)^T \\ &= kA^T \\ &= k(-A) \quad \text{since } A \text{ is skew-symmetric} \\ &= -(kA) \\ &= RS \end{aligned}$$

c) Prove that $A - A^T$ is skew symmetric.

To prove: $(A - A^T)^T = -(A - A^T)$

$$\begin{aligned} LS &= (A - A^T)^T \\ &= A^T - (A^T)^T \\ &= A^T - A \\ &= -(A - A^T) \\ &= RS \end{aligned}$$

d) Prove that $(A - A^T)^{-1}$ is skew symmetric if $A - A^T$ is invertible.

To prove: $\left[(A - A^T)^{-1}\right]^T = -(A - A^T)^{-1}$

$$\begin{aligned} LS &= \left[(A - A^T)^{-1}\right]^T \\ &= \left[(A - A^T)^T\right]^{-1} \\ &= \left[A^T - (A^T)^T\right]^{-1} \\ &= (A^T - A)^{-1} \\ &= \left(- (A - A^T)\right)^{-1} \\ &= - (A - A^T)^{-1} \\ &= RS \end{aligned}$$

3. Prove that if A and B are $n \times n$ matrices such that $A^2 = B^2 = (AB)^2 = I$ then $AB = BA$.

Since, since $A^2 = AA = I$ and $B^2 = BB = I$ then A and B are invertible with $A^{-1} = A$ and $B^{-1} = B$.

Since $(AB)^2 = (AB)(AB) = I$, then AB is invertible and $(AB)^{-1} = AB$.

To prove: $AB = BA$

$$LS = AB$$

$$= (AB)^{-1}$$

$$= B^{-1}A^{-1}$$

$$= BA$$

$$= RS$$

4. Let A be a matrix such that $A^3 = I$.

- a) Prove that A is invertible.

$$A^3 = I$$

$$\det(A^3) = \det(I)$$

$$[\det(A)]^3 = 1$$

$$\det(A) = 1$$

Thus, since $\det(A) \neq 0$, then A is invertible.

- b) Prove that $A^{-1} = A^2$

$$A^3 = I$$

$$A^3 A^{-1} = A^{-1} \quad \text{since } A \text{ is invertible}$$

$$A^2 = A^{-1}$$

5. Solve the following systems of linear equations, if possible.

a) $2x - y + 2z = 1$

$$x + y - 3z = 4$$

$$5x - y + z = 6$$

$$x + 4y - 11z = 11$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 1 & 1 & -3 & 4 \\ 5 & -1 & 1 & 6 \\ 1 & 4 & -11 & 11 \end{array} \right] \begin{array}{l} R_2 \rightarrow 2R_2 - R_1 \\ R_3 \rightarrow 2R_3 - 5R_1 \\ R_4 \rightarrow 2R_4 - R_1 \end{array} \left[\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 0 & 3 & -8 & 7 \\ 0 & 3 & -8 & 7 \\ 0 & 9 & -24 & 21 \end{array} \right]$$

$$\begin{array}{l} R_3 \rightarrow R_3 - R_2 \\ R_4 \rightarrow R_4 - 3R_2 \end{array} \left[\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 0 & 3 & -8 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} R_1 \rightarrow \frac{1}{2}R_3 \\ R_2 \rightarrow \frac{1}{3}R_2 \end{array} \left[\begin{array}{ccc|c} 1 & \frac{-1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & \frac{-8}{3} & \frac{7}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} c_3 = t \\ c_2 = \frac{7}{3} + \frac{8}{3}t \\ c_1 = \frac{5}{3} + \frac{1}{3}t \end{array}$$

Solution: $(\frac{5}{3} + \frac{1}{3}t, \frac{7}{3} + \frac{8}{3}t, t)$

b) $4x - y + 2z = 3$

$2x - 5y + z = 9$

$2x + 4y + z = -6$

$$\left[\begin{array}{ccc|c} 4 & -1 & 2 & 3 \\ 2 & -5 & 1 & 9 \\ 2 & 4 & 1 & -6 \end{array} \right] \begin{array}{l} R_2 \rightarrow 2R_2 - R_1 \\ R_3 \rightarrow 2R_3 - R_1 \end{array} \left[\begin{array}{ccc|c} 4 & -1 & 2 & 3 \\ 0 & -9 & 0 & 15 \\ 0 & 9 & 0 & -15 \end{array} \right]$$

$$\begin{array}{l} R_3 \rightarrow R_3 + R_2 \\ R_1 \rightarrow \frac{1}{4}R_1 \\ R_2 \rightarrow \frac{-1}{9}R_2 \end{array} \left[\begin{array}{ccc|c} 4 & -1 & 2 & 3 \\ 0 & -9 & 0 & 15 \\ 0 & 0 & 0 & 0 \end{array} \right] \left[\begin{array}{ccc|c} 1 & \frac{-1}{4} & \frac{1}{2} & \frac{3}{4} \\ 0 & 1 & 0 & \frac{-5}{3} \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} c_3 = t \\ c_2 = \frac{-5}{3} \\ c_1 = \frac{1}{3} - \frac{1}{2}t \end{array}$$

Solution: $(\frac{1}{3} - \frac{1}{2}t, -\frac{5}{3}, t)$

c) $2x - y + z - 5t = 12$

$6x + 3y - 2z + 3w + t = 1$

$2x + 5y - 4z + 3w + 11t = 8$

$$\left[\begin{array}{ccccc|c} 2 & -1 & 1 & 0 & -5 & 12 \\ 6 & 3 & -2 & 3 & 1 & 1 \\ 2 & 5 & -4 & 3 & 11 & 8 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array} \left[\begin{array}{ccccc|c} 2 & -1 & 1 & 0 & -5 & 12 \\ 0 & 6 & -5 & 3 & 16 & -35 \\ 0 & 6 & -5 & 3 & 16 & -4 \end{array} \right]$$

$$\begin{array}{l} R_3 \rightarrow R_3 - R_2 \\ R_1 \rightarrow \frac{1}{2}R_1 \\ R_2 \rightarrow \frac{1}{6}R_2 \\ R_3 \rightarrow \frac{1}{31}R_3 \end{array} \left[\begin{array}{ccccc|c} 2 & -1 & 1 & 0 & -5 & 12 \\ 0 & 6 & -5 & 3 & 16 & -35 \\ 0 & 0 & 0 & 0 & 0 & 31 \end{array} \right] \left[\begin{array}{ccccc|c} 1 & \frac{-1}{2} & \frac{1}{2} & 0 & \frac{-5}{2} & 6 \\ 0 & 1 & \frac{-5}{6} & \frac{1}{2} & \frac{8}{3} & \frac{-35}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right]$$

No solutions

d) $3x - y + 2z = 9$

$5x - y + 3z = 16$

$2x + 3y - z = 11$

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 9 \\ 5 & -1 & 3 & 16 \\ 2 & 3 & -1 & 11 \end{array} \right] \begin{array}{l} R_2 \rightarrow 3R_2 - 5R_1 \\ R_3 \rightarrow 3R_3 - 2R_1 \end{array} \left[\begin{array}{ccc|c} 3 & -1 & 2 & 9 \\ 0 & 2 & -1 & 3 \\ 0 & 11 & -7 & 15 \end{array} \right]$$

$$\begin{array}{l} R_3 \rightarrow 2R_3 - 11R_2 \\ R_1 \rightarrow \frac{1}{3}R_1 \\ R_2 \rightarrow \frac{1}{2}R_2 \\ R_3 \rightarrow \frac{-1}{3}R_3 \end{array} \left[\begin{array}{ccc|c} 3 & -1 & 2 & 9 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & -3 & -3 \end{array} \right] \left[\begin{array}{ccc|c} 1 & \frac{-1}{3} & \frac{2}{3} & 3 \\ 0 & 1 & \frac{-1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & 1 \end{array} \right]$$

Solution: $(3, 2, 1)$

6. Solve the following systems of linear equations using (i) Cramer's rule (ii) the inverse.

$$3x - y + z = 1$$

a) $5x + 3z = 14$

$$x + 3y - z = 4$$

i - Cramer's Rule

$$\det(A) = \begin{vmatrix} 3 & -1 & 0 \\ 5 & 0 & 3 \\ 1 & 3 & -1 \end{vmatrix} = 3 \begin{vmatrix} 0 & 3 \\ 3 & -1 \end{vmatrix} - (-1) \begin{vmatrix} 5 & 3 \\ 1 & -1 \end{vmatrix} + 0 = -27 - 8 = -35$$

$$\det(A(1)) = \begin{vmatrix} 1 & -1 & 0 \\ 14 & 0 & 3 \\ 4 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ 3 & -1 \end{vmatrix} - (-1) \begin{vmatrix} 14 & 3 \\ 4 & -1 \end{vmatrix} + 0 = -9 - 26 = -35$$

$$\det(A(2)) = \begin{vmatrix} 3 & 1 & 0 \\ 5 & 14 & 3 \\ 1 & 4 & -1 \end{vmatrix} = 3 \begin{vmatrix} 14 & 3 \\ 4 & -1 \end{vmatrix} - \begin{vmatrix} 5 & 3 \\ 1 & -1 \end{vmatrix} + 0 = -78 + 8 = -70$$

$$\det(A(3)) = \begin{vmatrix} 3 & -1 & 1 \\ 5 & 0 & 14 \\ 1 & 3 & 4 \end{vmatrix} = 3 \begin{vmatrix} 0 & 14 \\ 3 & 4 \end{vmatrix} - (-1) \begin{vmatrix} 5 & 14 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 5 & 0 \\ 1 & 3 \end{vmatrix} = -126 + 6 + 15 = -105$$

$$x = \frac{\det(A(1))}{\det(A)} = \frac{-35}{-35} = 1$$

$$y = \frac{\det(A(2))}{\det(A)} = \frac{-70}{-35} = 2$$

$$z = \frac{\det(A(3))}{\det(A)} = \frac{-105}{-35} = 3$$

Solution: (1, 2, 3)

ii - The inverse.

$$\text{cof}(A) = \begin{bmatrix} -9 & 8 & 15 \\ -1 & -3 & -10 \\ -3 & -9 & 5 \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} -9 & -1 & -3 \\ 8 & -3 & -9 \\ 15 & -10 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \begin{bmatrix} \frac{9}{35} & \frac{1}{35} & \frac{3}{35} \\ \frac{-8}{35} & \frac{3}{35} & \frac{9}{35} \\ \frac{-3}{7} & \frac{2}{7} & \frac{-1}{7} \end{bmatrix}$$

$$X = A^{-1}b = \begin{bmatrix} \frac{9}{35} & \frac{1}{35} & \frac{3}{35} \\ \frac{-8}{35} & \frac{3}{35} & \frac{9}{35} \\ \frac{-3}{7} & \frac{2}{7} & \frac{-1}{7} \end{bmatrix} \begin{bmatrix} 1 \\ 14 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$2x + y + 4z = 8$$

$$\text{b) } 2x - y + \quad = -16$$

$$3x \quad + 5z = 2$$

i - Cramer's Rule

$$\det(A) = \begin{vmatrix} 2 & 1 & 4 \\ 2 & -1 & 0 \\ 3 & 0 & 5 \end{vmatrix} = 3 \begin{vmatrix} 1 & 4 \\ -1 & 0 \end{vmatrix} - 0 + \begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix} = 12 - 4 = 8$$

$$\det(A(1)) = \begin{vmatrix} 8 & 1 & 4 \\ -16 & -1 & 0 \\ 2 & 0 & 5 \end{vmatrix} = 2 \begin{vmatrix} 1 & 4 \\ -1 & 0 \end{vmatrix} - 0 + \begin{vmatrix} 8 & 1 \\ -16 & -1 \end{vmatrix} = 8 + 8 = 16$$

$$\det(A(2)) = \begin{vmatrix} 2 & 8 & 4 \\ 2 & -16 & 0 \\ 3 & 2 & 5 \end{vmatrix} = -2 \begin{vmatrix} 8 & 4 \\ 2 & 5 \end{vmatrix} - 16 \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} + 0 = -64 + 32 = -32$$

$$\det(A(3)) = \begin{vmatrix} 2 & 1 & 8 \\ 2 & -1 & -16 \\ 3 & 0 & 2 \end{vmatrix} = 3 \begin{vmatrix} 1 & 8 \\ -1 & -16 \end{vmatrix} - 0 + 2 \begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix} = -24 - 8 = -32$$

$$x = \frac{\det(A(1))}{\det(A)} = \frac{16}{8} = 2$$

$$y = \frac{\det(A(2))}{\det(A)} = \frac{-32}{8} = -4$$

$$z = \frac{\det(A(3))}{\det(A)} = \frac{-32}{8} = -4$$

Solution: $(-6, 4, 4)$

ii - The inverse.

$$\text{cof}(A) = \begin{bmatrix} -5 & -10 & 3 \\ -5 & -2 & 3 \\ 4 & 8 & -4 \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} -5 & -5 & 4 \\ -10 & -2 & 8 \\ 3 & 3 & -4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \begin{bmatrix} \frac{5}{8} & \frac{5}{8} & \frac{-1}{2} \\ \frac{5}{4} & \frac{1}{4} & -1 \\ \frac{-3}{8} & \frac{-3}{8} & \frac{1}{2} \end{bmatrix}$$

$$X = A^{-1}b = \begin{bmatrix} \frac{5}{8} & \frac{5}{8} & \frac{-1}{2} \\ \frac{5}{4} & \frac{1}{4} & -1 \\ \frac{-3}{8} & \frac{-3}{8} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 8 \\ -16 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \\ 4 \end{bmatrix}$$

7. Consider the matrix $A = \begin{bmatrix} 3 & -1 \\ 5 & 4 \end{bmatrix}$.

a) Find the inverse of A using the adjoint.

$$\text{cof}(A) = \begin{bmatrix} 4 & -5 \\ 1 & 3 \end{bmatrix} \quad \text{adj}(A) = \begin{bmatrix} 4 & 1 \\ -5 & 3 \end{bmatrix} \quad \det(A) = 17$$

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \begin{bmatrix} \frac{4}{17} & \frac{1}{17} \\ \frac{-5}{17} & \frac{3}{17} \end{bmatrix}$$

b) Express A^{-1} as a product of elementary matrices.

$$\left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 5 & 4 & 0 & 1 \end{array} \right] \xrightarrow{R_1 \rightarrow \frac{1}{3}R_1} \left[\begin{array}{cc|cc} 1 & \frac{-1}{3} & \frac{1}{3} & 0 \\ 5 & 4 & 0 & 1 \end{array} \right] \quad E_1 = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \rightarrow R_2 - 5R_1} \left[\begin{array}{cc|cc} 1 & \frac{-1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{17}{3} & \frac{-5}{3} & 1 \end{array} \right] \quad E_2 = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \rightarrow \frac{3}{17}R_2} \left[\begin{array}{cc|cc} 1 & \frac{-1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{-5}{17} & \frac{3}{17} \end{array} \right] \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{17} \end{bmatrix}$$

$$\xrightarrow{R_1 \rightarrow R_1 + \frac{1}{3}R_2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{4}{17} & \frac{1}{17} \\ 0 & 1 & \frac{-5}{17} & \frac{3}{17} \end{array} \right] \quad E_4 = \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 1 \end{bmatrix}$$

$$A^{-1} = E_4 E_3 E_2 E_1$$

$$\begin{bmatrix} \frac{4}{17} & \frac{1}{17} \\ \frac{-5}{17} & \frac{3}{17} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{17} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$$

c) Express A as a product of elementary matrices.

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$$

$$\begin{bmatrix} 3 & -1 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{17}{3} \end{bmatrix} \begin{bmatrix} 1 & \frac{-1}{3} \\ 0 & 1 \end{bmatrix}$$

8. Consider the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 1 & 2 \\ 3 & 3 & 0 \end{bmatrix}$.

a) Find the inverse of A using the adjoint.

$$\text{cof}(A) = \begin{bmatrix} -6 & 6 & 9 \\ 3 & -3 & -6 \\ -1 & 0 & 2 \end{bmatrix} \quad \text{adj}(A) = \begin{bmatrix} -6 & 3 & -1 \\ 6 & -3 & 0 \\ 9 & -6 & 2 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 2 & 0 & 1 \\ 4 & 1 & 2 \\ 3 & 3 & 0 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} - 0 + \begin{vmatrix} 4 & 1 \\ 3 & 3 \end{vmatrix} = -12 + 9 = -3$$

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \begin{bmatrix} 2 & -1 & \frac{1}{3} \\ -2 & 1 & 0 \\ -3 & 2 & \frac{-2}{3} \end{bmatrix}$$

b) Express A^{-1} and A as a product of elementary matrices.

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 1 & 0 \\ 3 & 3 & 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_1 \rightarrow \frac{1}{2}R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 4 & 1 & 2 & 0 & 1 & 0 \\ 3 & 3 & 0 & 0 & 0 & 1 \end{array} \right] \quad E_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{l} R_2 \rightarrow R_2 - 4R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 3 & \frac{-3}{2} & \frac{-3}{2} & 0 & 1 \end{array} \right] \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$\underline{R_3 \rightarrow R_3 - 3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & \frac{-3}{2} & \frac{9}{2} & -3 & 1 \end{array} \right] \quad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$\underline{R_3 \rightarrow \frac{-2}{3}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 2 & \frac{-2}{3} \end{array} \right] \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{-2}{3} \end{bmatrix}$$

$$\underline{R_1 \rightarrow R_1 - \frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{2} & 2 & -1 & \frac{1}{3} \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 2 & \frac{-2}{3} \end{array} \right] \quad E_6 = \begin{bmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = E_6 E_5 E_4 E_3 E_2 E_1$$

$$\begin{bmatrix} 2 & -1 & \frac{1}{3} \\ -2 & 1 & 0 \\ -3 & 2 & \frac{-2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{-1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{-2}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1}$$

$$\begin{bmatrix} 2 & 0 & 1 \\ 4 & 1 & 2 \\ 3 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{-3}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

9. A coin bank has only nickels, dimes and quarters. The value of the coins is \$2. There are twice as many nickles as dimes and one more dime than quarters. Find the number of each coin in the bank.

$$5x + 10y + 25z = 200$$

$$x - 2y = 0$$

$$y - z = 1$$

$$\left[\begin{array}{ccc|c} 5 & 10 & 25 & 200 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{array} \right] \xrightarrow{R_2 \rightarrow 5R_2 - R_1} \left[\begin{array}{ccc|c} 5 & 10 & 25 & 200 \\ 0 & -20 & -25 & -200 \\ 0 & 1 & -1 & 1 \end{array} \right]$$

$$\xrightarrow{R_3 \rightarrow 20R_3 + R_2} \left[\begin{array}{ccc|c} 5 & 10 & 25 & 200 \\ 0 & -20 & -25 & -200 \\ 0 & 0 & -45 & -180 \end{array} \right] \begin{array}{l} R_1 \rightarrow \frac{1}{5}R_1 \\ R_2 \rightarrow \frac{-1}{20}R_2 \\ R_3 \rightarrow \frac{-1}{45}R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 5 & 40 \\ 0 & 1 & \frac{5}{4} & 10 \\ 0 & 0 & 1 & 4 \end{array} \right]$$

$$z = 4, y = 5, x = 10$$

Thus there are 10 nickels, 5 dimes and 4 quarters in the bank.

10. Suppose a man has three modes of transportation to work: he can walk, drive his car, or take the bus. If he walks one day, then he will either take the car the next day with a probability of $\frac{2}{3}$ or take the bus with a probability of $\frac{1}{3}$. If he drove one day, then he will walk the next day with a probability of $\frac{1}{2}$ and take the bus with a probability of $\frac{1}{2}$. If he took the bus one day, then he will walk the next day with a probability of $\frac{1}{3}$, drive with a probability of $\frac{1}{3}$ and take the bus with a probability of $\frac{1}{3}$.

a) Find the transition matrix. $A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$

- b) If he drives on Monday, find the probability that he will walk, drive or take the bus on Wednesday.

$$X_1 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}$$

Walk: $\frac{1}{6}$ Drive: $\frac{1}{2}$ Bus: $\frac{1}{3}$

- c) In the long run, how often will he walk, drive and take the bus?

$$(I - A)X = 0$$

$$\left[\begin{array}{ccc|c} 1 & \frac{-1}{2} & \frac{-1}{3} & 0 \\ \frac{-2}{3} & 1 & \frac{-1}{3} & 0 \\ \frac{-1}{3} & \frac{-1}{2} & \frac{2}{3} & 0 \end{array} \right] \begin{array}{l} R_2 \rightarrow 3R_2 + 2R_1 \\ R_3 \rightarrow 3R_3 + R_1 \end{array} \left[\begin{array}{ccc|c} 1 & \frac{-1}{2} & \frac{-1}{3} & 0 \\ 0 & 2 & \frac{-5}{3} & 0 \\ 0 & -2 & \frac{5}{3} & 0 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 + R_2} \left[\begin{array}{ccc|c} 1 & \frac{-1}{2} & \frac{-1}{3} & 0 \\ 0 & 2 & \frac{-5}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\underline{R_2 \rightarrow \frac{1}{2}R_2} \begin{bmatrix} 1 & \frac{-1}{2} & \frac{-1}{3} & 0 \\ 0 & 1 & \frac{-5}{6} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} z = t \\ y = \frac{5}{6}t \\ x = \frac{3}{4}t \end{array}$$

$$\frac{3}{4}t + \frac{5}{6}t + t = 1$$

$$t = \frac{12}{31}$$

$$\text{Walk: } \frac{9}{31}$$

$$\text{Drive: } \frac{10}{31}$$

$$\text{Bus: } \frac{12}{31}$$

$$\text{Thus } X = \left(\frac{9}{31}, \frac{10}{31}, \frac{12}{31} \right)$$

11. Find the equation of the parabola passing through the points $A(1,4)$, $B(2,12)$ and $C(-3,32)$.

$$y = a + bx + cx^2$$

$$4 = 1 + x + x^2$$

$$12 = 1 + 2x + 4x^2$$

$$32 = 1 - 3x + 9x^2$$

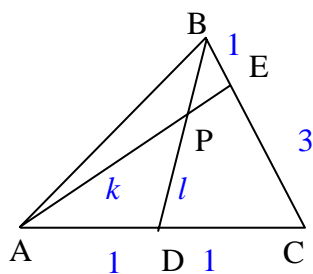
$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 12 \\ 1 & -3 & 9 & 32 \end{bmatrix} \xrightarrow{\substack{R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1}} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 8 \\ 0 & -4 & 8 & 28 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 + 4R_2} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 8 \\ 0 & 0 & 20 & 60 \end{bmatrix}$$

$$\underline{R_3 \rightarrow \frac{1}{20}R_3} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 8 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad \begin{array}{l} c = 3 \\ b = -1 \\ a = 2 \end{array}$$

Thus the parabola is $y = 2 - x + 3x^2$.

12. Let ABC be a triangle and E a point on the segment BC dividing it in a ratio of 1 to 3. Let D be the midpoint of AC . Join A to E and B to D , and let P be the point on intersection of the segments AE and BD . In what ratio does P divide AE and BD ?

Let $\overline{AP} = k\overline{AG}$ and $\overline{FP} = l\overline{FE}$.



We have

$$\overline{BE} = \frac{1}{4}\overline{BC}$$

$$\overline{AD} = \frac{1}{2}\overline{AC}$$

Let us express \overline{AP} in terms of \overline{AB} and \overline{AC} in two different ways.

$$\begin{aligned}
 \overrightarrow{AP} &= k\overrightarrow{AE} & \overrightarrow{AP} &= \overrightarrow{AD} + \overrightarrow{DP} \\
 &= k(\overrightarrow{AB} + \overrightarrow{BE}) & &= \frac{1}{2}\overrightarrow{AC} + l\overrightarrow{DB} \\
 &= k\overrightarrow{AB} + \frac{1}{4}k\overrightarrow{BC} & &= \frac{1}{2}\overrightarrow{AC} + l(\overrightarrow{DA} + \overrightarrow{AB}) \\
 &= k\overrightarrow{AB} + \frac{k}{4}(\overrightarrow{BA} + \overrightarrow{AC}) & &= \frac{1}{2}\overrightarrow{AC} + l\left(\frac{-1}{2}\overrightarrow{AC} + \overrightarrow{AB}\right) \\
 &= \frac{3k}{4}\overrightarrow{AB} + \frac{k}{4}\overrightarrow{AC} & &= \left(\frac{1}{2} - \frac{1}{2}l\right)\overrightarrow{AC} + l\overrightarrow{AB}
 \end{aligned}$$

By the basis theorem, we have the equations

$$\frac{3k}{4} = l \quad \frac{1}{4}k = \frac{1}{2} - \frac{1}{2}l$$

Combining these equations, we have

$$\frac{1}{4}k = \frac{1}{2} - \frac{1}{2}\frac{3k}{4}$$

$$\frac{5}{8}k = \frac{1}{2}$$

$$k = \frac{4}{5}$$

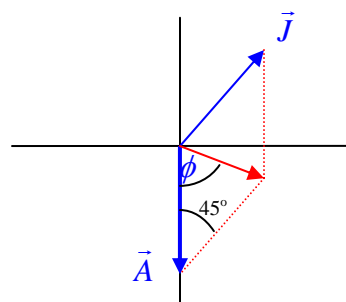
thus $k = \frac{4}{5}$ and $l = \frac{3}{5}$.

Hence, P divides AE in a ration of **4 to 1** and divides DB in a ration of **3 to 2**.

13. Let ABC be a triangle and M, N and P the midpoints of AB, BC and CA respectively. Prove that if O is any point (inside or outside the triangle) then

$$\begin{aligned}
 \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} &= \overrightarrow{OM} + \overrightarrow{ON} + \overrightarrow{OP} \\
 \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} &= (\overrightarrow{OM} + \overrightarrow{MA}) + (\overrightarrow{ON} + \overrightarrow{NB}) + (\overrightarrow{OP} + \overrightarrow{PC}) \\
 &= \overrightarrow{OM} + \overrightarrow{ON} + \overrightarrow{OP} + \overrightarrow{MA} + \overrightarrow{NB} + \overrightarrow{PC} \\
 &= \overrightarrow{OM} + \overrightarrow{ON} + \overrightarrow{OP} + \frac{1}{2}\overrightarrow{BA} + \frac{1}{2}\overrightarrow{CB} + \frac{1}{2}\overrightarrow{AC} \\
 &= \overrightarrow{OM} + \overrightarrow{ON} + \overrightarrow{OP} + \frac{1}{2}(\overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BA}) \\
 &= \overrightarrow{OM} + \overrightarrow{ON} + \overrightarrow{OP} + \frac{1}{2}\vec{0} \\
 &= \overrightarrow{OM} + \overrightarrow{ON} + \overrightarrow{OP}
 \end{aligned}$$

14. A Boeing 737 aircraft maintains a constant airspeed of 500 miles per hour in the direction due south. The velocity of the jet stream is 80 miles per hour in a northeasterly direction (N45°E). Find the actual speed and direction of the aircraft relative to the ground.



$$\begin{aligned}
 \|\vec{A} + \vec{J}\|^2 &= \|\vec{A}\|^2 + \|\vec{J}\|^2 + 2\|\vec{A}\|\|\vec{J}\|\cos\theta \\
 &= 500^2 + 80^2 - 2 \cdot 500 \cdot 80 \cos 45^\circ \\
 &= 199831
 \end{aligned}$$

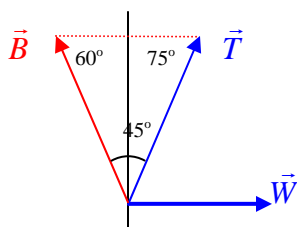
$$\|\vec{A} + \vec{J}\| = 447$$

$$\frac{\sin 45^\circ}{447} = \frac{\sin \phi}{80} \quad \phi = 7.3^\circ$$

Thus the plane has a speed of 447 miles per hour in a S7.3°E direction.

15. A river flows from west to east. There are ferry terminals on the north and south shores, the north dock being 15° east of north (i.e. $N15^\circ E$) from the south dock. The ferry captain knows from experience that in order to reach the dock on the north shore from the south shore dock, she has to steer $N30^\circ W$.

a) If the ferry travels at 12 km/h, what is the speed of the current?



$$\frac{\sin 45^\circ}{\|\vec{W}\|} = \frac{\sin 75^\circ}{12}$$

$$\|\vec{W}\| = 8.78$$

Speed of current is 8.78 km/h

b) If the trip takes $\frac{1}{4}$ hour, how far apart are the dock?

$$\frac{\sin 60^\circ}{\|\vec{T}\|} = \frac{\sin 75^\circ}{12}$$

$$\|\vec{T}\| = 10.76 \text{ km/h}$$

Thus the distance between the docks is $\frac{1}{4}10.76 = 2.69 \text{ km}$

16. Consider the vectors $\vec{u} = (-2, 5, 5)$, $\vec{v} = (1, -1, 2)$ and $\vec{w} = (5, -1, 2)$.

a) Evaluate $2\vec{u} - 3\vec{v}$

$$2\vec{u} - 3\vec{v} = 2(-2, 5, 5) - 3(1, -1, 2) = (-7, 13, 4)$$

b) Find the vector projection of \vec{u} onto \vec{w} .

$$\text{proj}_{\vec{w}} \vec{u} = \frac{\vec{u} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w} = \frac{-5}{30} (5, -1, 2) = \left(-\frac{5}{6}, \frac{1}{6}, \frac{-1}{3} \right)$$

c) Find the angle between \vec{u} and \vec{w} .

$$\cos \theta = \frac{\vec{u} \cdot \vec{w}}{\|\vec{u}\| \|\vec{w}\|} = \frac{-5}{\sqrt{54} \sqrt{30}} \quad \theta \approx 97.1^\circ$$

d) Find the area of the triangle having sides \vec{u} and \vec{v} .

$$A = \frac{1}{2} \|\vec{u} \times \vec{v}\| = \frac{1}{2} \begin{vmatrix} i & j & k \\ -2 & 5 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \frac{1}{2} \|(15, 9, -3)\| = \frac{3\sqrt{35}}{2}$$

e) Find the volume of the parallelepiped having sides \vec{u} , \vec{v} and \vec{w} .

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})| = \begin{vmatrix} -2 & 5 & 5 \\ 1 & -1 & 2 \\ 5 & -1 & 2 \end{vmatrix} = \begin{vmatrix} -2 & -1 & 2 \\ -1 & 2 & 2 \end{vmatrix} - 5 \begin{vmatrix} 1 & 2 \\ 5 & 2 \end{vmatrix} + 5 \begin{vmatrix} 1 & -1 \\ 5 & -1 \end{vmatrix} = |0 + 40 + 20| = 60$$

17. Consider the points $A(2, -1, 3)$, and $B(3, -1, 5)$, $C(-2, 2, 3)$ and $D(-1, 0, 5)$.

a) Find the vector projection of \vec{AB} onto \vec{AC} .

$$\text{proj}_{\vec{AC}} \vec{AB} = \frac{\vec{AB} \cdot \vec{AC}}{\vec{AC} \cdot \vec{AC}} \vec{AC} = \frac{(1, 0, 2) \cdot (-4, 3, 0)}{(-4, 3, 0) \cdot (-4, 3, 0)} (-4, 3, 0) = \frac{-4}{25} (-4, 3, 0) = \left(\frac{16}{25}, \frac{-12}{25}, 0 \right)$$

- b) Find the angle between \overrightarrow{AB} and \overrightarrow{AC} .

$$\cos \theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\|\overrightarrow{AB}\| \|\overrightarrow{AC}\|} = \frac{-4}{\sqrt{5}\sqrt{25}} \quad \theta \approx 111$$

- c) Find the volume of the tetrahedron ABCD.

$$V = \frac{1}{6} \left| \overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) \right| = \frac{1}{6} \left| \begin{vmatrix} 1 & 0 & 2 \\ -4 & 3 & 0 \\ -3 & 1 & 2 \end{vmatrix} \right| = \frac{1}{6} \left| 1 \begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} - 0 + 2 \begin{vmatrix} -4 & 3 \\ -3 & 1 \end{vmatrix} \right| = \frac{1}{6} |6 + 10| = \frac{8}{3}$$

- d) Find the equation of the line (in **parametric form**) passing through D and parallel to \overrightarrow{AB} .

$$\vec{u} = \overrightarrow{AB} = (1, 0, 2) \quad l: \begin{cases} x = -1 + t \\ y = 0 \\ z = 5 + 2t \end{cases}$$

- e) Find the equation of the plane (in **general form**) parallel to \overrightarrow{AB} and \overrightarrow{AC} , and passing through D .

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ -4 & 3 & 0 \end{vmatrix} = (-6, -8, 3)$$

$$-6x - 8y + 3z = -6(-1) - 8(0) + 3(5) = 21$$

$$\pi: 6x + 8y - 3z = -21$$

- f) Find the equation of the plane perpendicular to \overrightarrow{AC} and passing through D .

$$\vec{n} = \overrightarrow{AC} = (-4, 3, 0) \quad -4x + 3y = -4(-1) + 3(0) + 0(5) = 4$$

$$\pi: 4x - 3y = -4$$

18. Consider the plane $\pi: 2x + y - 5z + 1 = 0$ and the line $L: \frac{x-1}{3} = \frac{2y+1}{4} = 3-z$

- a) Find the equation of the line (in **symmetric form**) perpendicular to π and passing through $P(1,1,-3)$.

$$\vec{u} = (2, 1, -5) \quad \frac{x-1}{2} = y-1 = \frac{z+3}{-5}$$

- b) Find the equation of the line (in **parametric form**) parallel to L and passing through $P(1,1,-3)$.

$$\vec{u} = (3, 2, -1) \quad \begin{cases} x = 1 + 3t \\ y = 1 + 2t \\ z = -3 - t \end{cases}$$

- c) Find the distance between the line L and the line found in (b).

Since the lines are parallel, we have

$$\begin{array}{l} P_1(1, \frac{-1}{2}, 3) \\ P_2(1, 1, -3) \end{array} \quad \overline{P_1P_2} = (0, \frac{3}{2}, -6)$$

$$d = \frac{\|\overline{P_1P_2} \times \vec{u}\|}{\|\vec{u}\|} = \frac{\left\| \begin{vmatrix} i & j & k \\ 0 & \frac{3}{2} & -6 \\ 3 & 2 & -1 \end{vmatrix} \right\|}{\sqrt{14}} = \frac{\left\| (\frac{21}{2}, -18, \frac{9}{2}) \right\|}{\sqrt{14}} = \frac{\frac{3}{2}\sqrt{202}}{\sqrt{14}} = \frac{3}{14}\sqrt{707}$$

- d) Find the intersection, if possible, of the plane π and the line L .

$$L := \begin{cases} x = 1 + 3t & 2(1 + 3t) + (\frac{-1}{2} + 2t) - 5(3 - t) + 1 = 0 \\ y = \frac{-1}{2} + 2t & 13t = \frac{25}{2} \\ z = 3 - t & t = \frac{25}{26} \end{cases}$$

$$\text{Intersection: } (1 + 3\frac{25}{26}, \frac{-1}{2} + 2\frac{25}{26}, 3 - \frac{25}{26}) = (\frac{101}{26}, \frac{37}{26}, \frac{53}{26})$$

- e) Find the equation of the plane π in vector form.

$$\begin{array}{l} z = t \\ y = s \\ x = -\frac{1}{2} - \frac{1}{2}s + \frac{5}{2}t \end{array} \quad \pi : (x, y, z) = (\frac{-1}{2}, 0, 0) + s(\frac{-1}{2}, 1, 0) + t(\frac{5}{2}, 0, 1)$$

- f) Find the equation of the plane π_2 (in **general form**) perpendicular to L and passing through $P(1, 1, -3)$.

$$\begin{array}{l} \vec{n}_2 = (3, 2, -1) \\ \pi_2 : 3x + 2y - z = 8 \end{array} \quad 3x + 2y - z = 3(1) + 2(1) - (-3) = 8$$

- g) Find the angle between the plane π and the plane π_2 found in (f).

$$\cos \theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\| \|\vec{n}_2\|} = \frac{|13|}{\sqrt{30}\sqrt{14}} \quad \theta \approx 50.6^\circ$$

- h) Find the point Q on the plane π that is closest to the point $P(1, 1, -3)$.

$$\begin{array}{l} R(0, -1, 0) \\ \overline{PR} = (-1, -2, 3) \\ \vec{n} = (2, 1, -5) \end{array} \quad \begin{array}{l} \overline{PQ} = \text{proj}_{\vec{n}} \overline{PR} = \frac{\overline{PR} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n} \\ (x-1, y-1, z+3) = \frac{-19}{30}(2, 1, -5) \\ (x, y, z) = \frac{-19}{30}(2, 1, -5) + (1, 1, -3) = (\frac{-4}{15}, \frac{11}{30}, \frac{1}{6}) \end{array}$$

$$Q(\frac{-4}{15}, \frac{11}{30}, \frac{1}{6})$$

- i) Find the point Q on the line L that is closest to the point $P(1,1,-3)$.

$$\begin{aligned} R(1, \frac{-1}{2}, 3) & \qquad \overline{RQ} = \text{proj}_{\vec{u}} \overline{RP} = \frac{\overline{PR} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} \\ \overline{PR} &= (0, \frac{-3}{2}, 6) & (x-1, y-1, z+3) &= \frac{-9}{14}(3, 2, -1) \\ \vec{u} &= (3, 2, -1) & (x, y, z) &= \frac{-9}{14}(3, 2, -1) + (1, 1, -3) = (\frac{41}{14}, \frac{11}{14}, \frac{33}{14}) \\ Q &= (\frac{41}{14}, \frac{11}{14}, \frac{33}{14}) \end{aligned}$$

- j) Find the distance from the point $P(1,1,-3)$ to the plane π .

$$\begin{aligned} R(0, -1, 0) & \qquad d = \frac{|\overline{PR} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{|-19|}{\sqrt{30}} = \frac{19\sqrt{30}}{30} \\ \overline{PR} &= (-1, -2, 3) \end{aligned}$$

- k) Find the distance from the point $P(1,1,-3)$ to the line L .

$$\begin{aligned} R(1, \frac{-1}{2}, 3) & \qquad \overline{PR} \times \vec{u} = \begin{vmatrix} i & j & k \\ 0 & \frac{-3}{2} & 6 \\ 3 & 2 & -1 \end{vmatrix} = (\frac{-21}{2}, 18, \frac{9}{2}) \\ \overline{PR} &= (0, \frac{-3}{2}, 6) \\ d &= \frac{\|\overline{PR} \times \vec{u}\|}{\|\vec{u}\|} = \frac{\|(\frac{-21}{2}, 18, \frac{9}{2})\|}{\sqrt{14}} = \frac{\frac{3}{2}\sqrt{202}}{\sqrt{14}} = \frac{3\sqrt{707}}{14} \end{aligned}$$

- l) Find the equation of the plane (if possible), in general form containing the lines L and

$$L_2 : \frac{x+5}{3} = \frac{y-2}{2} = -z.$$

$$\begin{aligned} \vec{u} &= (3, 2, -1) \\ \vec{u}_2 &= (3, 2, -1) \end{aligned} \qquad \therefore L \parallel L_2$$

$$P(1, \frac{-1}{2}, 3) \in L, \text{ Since } \frac{1+5}{3} \neq \frac{\frac{-1}{2}-2}{2} \neq -3 \text{ then } P \notin L_2$$

Thus L and L_2 are parallel and distinct.

$$\begin{aligned} P_2(-5, 2, 0) & \qquad \vec{n} = \vec{u} \times \overline{PP_2} = \begin{vmatrix} i & j & k \\ 3 & 2 & -1 \\ -6 & \frac{5}{2} & -3 \end{vmatrix} = (\frac{-7}{2}, 15, \frac{39}{2}) \\ \overline{PP_2} &= (-6, \frac{5}{2}, -3) \end{aligned}$$

$$\frac{-7}{2}x + 15y - \frac{39}{2}z = \frac{-7}{2}(-5) + 15(2) - 0 = \frac{95}{2} \qquad \text{Plane: } 7x - 30y - 39z = -95$$

m) Find the equation of the plane (if possible), in general form containing the lines L and

$$L_3: \frac{x+4}{2} = y + \frac{5}{2} = \frac{z+10}{3}.$$

$$\begin{aligned} \vec{u} &= (3, 2, -1) \\ \vec{u}_3 &= (2, 1, 3) \end{aligned} \quad \therefore L \not\parallel L_3$$

To find $L \cap L_3$, we have

$$L: (x, y, z) = (1, \frac{-1}{2}, 3) + t(3, 2, -1)$$

$$L_3: (x, y, z) = (-4, \frac{-5}{2}, -10) + s(2, 1, 3)$$

$$(1, \frac{-1}{2}, 3) + t(3, 2, -1) = (-4, \frac{-5}{2}, -10) + s(2, 1, 3)$$

$$t(3, 2, -1) + s(-2, -1, -3) = (-5, -2, -13)$$

$$\left[\begin{array}{ccc|c} 3 & -2 & -5 & \\ 2 & -1 & -2 & \\ -1 & -3 & -13 & \end{array} \right] \begin{array}{l} R_2 \rightarrow 3R_2 - 2R_1 \\ R_3 \rightarrow 3R_3 + R_1 \end{array} \left[\begin{array}{ccc|c} 3 & -2 & -5 & \\ 0 & 1 & 4 & \\ 0 & -11 & -44 & \end{array} \right] \begin{array}{l} R_3 \rightarrow 11R_3 + R_2 \end{array} \left[\begin{array}{ccc|c} 3 & -2 & -5 & \\ 0 & 1 & 4 & \\ 0 & 0 & 0 & \end{array} \right]$$

$$\begin{array}{l} R_1 \rightarrow \frac{1}{3}R_1 \\ \left[\begin{array}{ccc|c} 1 & \frac{-2}{3} & \frac{-5}{3} & \\ 0 & 1 & 4 & \\ 0 & 0 & 0 & \end{array} \right] \end{array} \quad \begin{array}{l} s = 4 \\ t = 1 \end{array}$$

$$(x, y, z) = (1, \frac{-1}{2}, 3) + (3, 2, -1) = (4, \frac{3}{2}, 2)$$

Thus $L \cap L_3 = P(4, \frac{3}{2}, 2)$, so the lines are nonparallel and intersecting.

$$\vec{n} = \vec{u} \times \vec{u}_3 = \begin{vmatrix} i & j & k \\ 3 & 2 & -1 \\ 2 & 1 & 3 \end{vmatrix} = (7, -11, -1) \quad 7x - 11y - z = 7(4) - 11(\frac{3}{2}) - 2 = \frac{19}{2}$$

$$\text{Plane: } 7x - 11y - z = \frac{19}{2}$$

19. Is the set V a vector space with the following operations?

$$\text{a) } V = \mathbb{R}^2 \quad (u_1, u_2) \oplus (v_1, v_2) = (u_1 - v_1, u_2 - v_2)$$

$$k \odot (u_1, u_2) = (ku_1, ku_2)$$

No, Axiom 2 fails: Counter example with $\vec{u} = (1, 2)$ and $\vec{v} = (3, 4)$

$$(1, 2) \oplus (3, 4) = (1 - 3, 2 - 4) = (-2, -2)$$

$$(3, 4) \oplus (1, 2) = (3 - 1, 4 - 2) = (2, 2)$$

Thus $(1, 2) \oplus (3, 4) \neq (3, 4) \oplus (1, 2)$, that is $\vec{u} \oplus \vec{v} \neq \vec{v} \oplus \vec{u}$.

$$\begin{aligned} \text{b) } V = \mathbb{R}^2 \quad (u_1, u_2) \oplus (v_1, v_2) &= (u_1 + v_1 + 1, u_2 + v_2 + 1) \\ k \odot (u_1, u_2) &= (k + ku_1 - 1, k + ku_2 - 1) \end{aligned}$$

Yes. Let $\vec{u} = (u_1, u_2)$, $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$ be in \mathbb{R}^2 .

1. $\vec{u} \oplus \vec{v} = (u_1 + v_1 + 1, u_2 + v_2 + 1) \in \mathbb{R}^2$
2. $\vec{u} \oplus \vec{v} = (u_1 + v_1 + 1, u_2 + v_2 + 1) = (v_1 + u_1 + 1, v_2 + u_2 + 1) = \vec{v} \oplus \vec{u}$
3. $\begin{aligned} \vec{u} \oplus (\vec{v} \oplus \vec{w}) &= (u_1, u_2) \oplus (v_1 + w_1 + 1, v_2 + w_2 + 1) \\ &= (u_1 + (v_1 + w_1 + 1) + 1, u_2 + (v_2 + w_2 + 1) + 1) \\ &= ((u_1 + v_1 + 1) + w_1 + 1, (u_2 + v_2 + 1) + w_2 + 1) \\ &= (u_1 + v_1 + 1, u_2 + v_2 + 1) \oplus (w_1, w_2) \\ &= (\vec{u} \oplus \vec{v}) \oplus \vec{w} \end{aligned}$
4. $\begin{aligned} \vec{0} = 0 \odot (u_1, u_2) &= (-1, -1) \\ \vec{u} \oplus \vec{0} &= (u_1, u_2) \oplus (-1, -1) = (u_1 - 1 + 1, u_2 - 1 + 1) = (u_1, u_2) = \vec{u} \\ -\vec{u} &= (-1) \odot (u_1, u_2) = (-1 - u_1 - 1, -1 - u_2 - 1) = (-2 - u_1, -2 - u_2) \\ \vec{u} \oplus -\vec{u} &= (u_1, u_2) \oplus (-2 - u_1, -2 - u_2) \\ &= (u_1 - 2 - u_1 + 1, u_2 - 2 - u_2 + 1) = (-1, -1) = \vec{0} \end{aligned}$
6. $k \odot (u_1, u_2) = (k + ku_1 - 1, k + ku_2 - 1) \in \mathbb{R}^2$
7. $\begin{aligned} (k+l) \odot \vec{u} &= (k+l) \odot (u_1, u_2) = ((k+l) + (k+l)u_1 - 1, (k+l) + (k+l)u_2 - 1) \\ &= ((k + ku_1 - 1) + (l + lu_1 - 1) + 1, (k + ku_2 - 1) + (l + lu_2 - 1) + 1) \\ &= (k + ku_1 - 1, k + ku_2 - 1) \oplus (l + lu_1 - 1, l + lu_2 - 1) \\ &= (k \odot \vec{u}) \oplus (l \odot \vec{u}) \end{aligned}$
8. $\begin{aligned} k \odot (\vec{u} \oplus \vec{v}) &= k \odot (u_1 + v_1 + 1, u_2 + v_2 + 1) \\ &= (k + k(u_1 + v_1 + 1) - 1, k + k(u_2 + v_2 + 1) - 1) \\ &= ((k + ku_1 - 1) + (k + kv_1 - 1) + 1, (k + ku_2 - 1) + (k + kv_2 - 1) + 1) \\ &= (k + ku_1 - 1, k + ku_2 - 1) \oplus (k + kv_1 - 1, k + kv_2 - 1) \\ &= (k \odot \vec{u}) \oplus (k \odot \vec{v}) \end{aligned}$
9. $\begin{aligned} (kl) \odot \vec{u} &= (kl + klu_1 - 1, kl + klu_2 - 1) \\ &= (k + k(l + lu_1 - 1) - 1, k + k(l + lu_2 - 1) - 1) \\ &= k \odot (l + lu_1 - 1, l + lu_2 - 1) \\ &= k \odot (l \odot \vec{u}) \end{aligned}$
10. $1 \odot \vec{u} = 1 \odot (u_1, u_2) = (1 + u_1 - 1, 1 + u_2 - 1) = (u_1, u_2) = \vec{u}$

20. Is the set V with standard operations a vector space?

a) $V = \{A : A \text{ is symmetric, } A \in M_{2,2}\}$

Yes V is nonempty since $0 \in V$, $0^T = 0$.

Let $A, B \in V$. Then $A^T = A$ and $B^T = B$.

1. $A+B \in V$ since $(A+B)^T = A^T + B^T = A+B$

2. $kA \in V$ since $(kA)^T = kA^T = kA$

b) $V = \{A : A \text{ is nilpotent, } A \in M_{2,2}\}$ (A is *nilpotent* if $A^2 = 0$)

No. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then $A^2 = 0$ and $B^2 = 0$, so $A, B \in V$ but

$$A+B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \notin V \text{ since } (A+B)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \neq 0$$

c) $V = \{A : A \text{ is lower triangular, } A \in M_{2,2}\}$

Yes V is nonempty since $0 \in V$.

Let $A = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \in V$ and $B = \begin{bmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{bmatrix} \in V$

1. $A+B = \begin{bmatrix} a_{11}+b_{11} & 0 \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} \in V$

2. $kA = \begin{bmatrix} ka_{11} & 0 \\ ka_{21} & ka_{22} \end{bmatrix} \in V$

d) $V = \{p(x) : p(1) = p(2), p(x) \in P_2\}$.

Yes V is nonempty since $p(x) = 0 \in V$ since $p(1) = 0 = p(2)$.

Let $p(x), q(x) \in V$. Then $p(1) = p(2)$ and $q(1) = q(2)$.

1. $(p+q)(x) \in V$ since $(p+q)(1) = p(1) + q(1) = p(2) + q(2) = (p+q)(2)$

2. $(kp)(x) \in V$ since $(kp)(1) = kp(1) = kp(2) = (kp)(2)$

21. For each of the subsets W in \mathbb{R}^3 ,

i) Is W a subspace of \mathbb{R}^3 ?

ii) Give a geometrical interpretation of W .

a) $W = \{(x, y, z) : 2x - 3y + z = 0, x, y, z \in \mathbb{R}\}$

Yes W is nonempty since $\vec{0} \in W$

Let $\vec{u} = (x_1, y_1, z_1) \in W$ and $\vec{v} = (x_2, y_2, z_2) \in W$.

Then $2x_1 - 3y_1 + z_1 = 0$ and $2x_2 - 3y_2 + z_2 = 0$

1. $\vec{u} + \vec{v} = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \in W$

$$\begin{aligned} \text{since } 2(x_1 + x_2) - 3(y_1 + y_2) + (z_1 + z_2) &= (2x_1 - 3y_1 + z_1) + (2x_2 - 3y_2 + z_2) \\ &= 0 + 0 = 0 \end{aligned}$$

$$2. \quad k\vec{u} = (kx_1, ky_1, kz_1) \in W \quad \text{since } 2(kx_1) - 3(ky_1) + kz_1 = k(2x_1 - 3y_1 + z_1) = k \cdot 0 = 0$$

Geometrically, W is the plane $2x - 3y + z = 0$.

$$\text{b) } W = \{(x, y, z) : 2x - 3y + z - 8 = 0, \quad x, y, z \in \mathbb{R}\}$$

No. If $\vec{u} = (4, 0, 0) \in W$ and $k = 0$,

$$\text{then } k\vec{u} = (0, 0, 0) \notin W \quad \text{since } 2 \cdot 0 - 3 \cdot 0 + 0 - 8 \neq 0$$

Geometrically, W is the plane $2x - 3y + z = 8$.

$$\text{c) } W = \{(a, 3a - 4, a + 1) : a \in \mathbb{R}\}$$

No. If $\vec{u} = (0, -4, 1) \in W$ and $k = 0$,

then $k\vec{u} = (0, 0, 0) \notin W$ since we would need to have

$$a = 0 \quad 3a - 4 = 0 \quad a + 1 = 0 \quad \text{which is impossible.}$$

$$a = \frac{4}{3} \quad a = -1$$

Geometrically, W is the line $x = \frac{y+4}{3} = z - 1$.

$$\text{d) } W = \{(2a - b + c, a + b + c, 3a + 2c) : a, b, c \in \mathbb{R}\}$$

Yes W is nonempty since $\vec{0} \in W$

Let $\vec{u} = (2a_1 - b_1 + c_1, a_1 + b_1 + c_1, 3a_1 + 2c_1) \in W$ and

$$\vec{v} = (2a_2 - b_2 + c_2, a_2 + b_2 + c_2, 3a_2 + 2c_2) \in W \quad \vec{v} = (x_2, y_2, z_2) \in W.$$

$$1. \quad \vec{u} + \vec{v} = ((2a_1 - b_1 + c_1) + (2a_2 - b_2 + c_2), (a_1 + b_1 + c_1) + (a_2 + b_2 + c_2), (3a_1 + 2c_1) + (3a_2 + 2c_2))$$

$$= ((a_1 + a_2) - (b_1 + b_2) + (c_1 + c_2), (a_1 + a_2) + (b_1 + b_2) + (c_1 + c_2),$$

$$3(a_1 + a_2) + 2(c_1 + c_2)) \in W$$

$$2. \quad k\vec{u} = (k(2a_1 - b_1 + c_1), k(a_1 + b_1 + c_1), k(3a_1 + 2c_1))$$

$$= (2(ka_1) - (kb_1) + (kc_1), ka_1 + kb_1 + kc_1, 3(ka_1) + 2(kc_1)) \in W$$

To give a geometrical description of W , we need to find a basis for W .

Since $(2a - b + c, a + b + c, 3a + 2c) = a(2, 1, 3) + b(-1, 1, 0) + c(1, 1, 2)$

then if $B = \{(2, 1, 3), (-1, 1, 0), (1, 1, 2)\}$ we have $W = \text{span}(B)$.

$$c_1(2, 1, 3) + c_2(-1, 1, 0) + c_3(1, 1, 2) = (0, 0, 0)$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 3 & 0 & 2 & 0 \end{array} \right] \begin{array}{l} R_2 \rightarrow 2R_2 - R_1 \\ R_3 \rightarrow 2R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 3 & 1 & 0 \end{array} \right]$$

$$\underline{R_3 \rightarrow R_3 - R_2} \begin{bmatrix} 2 & -1 & 1 & | & 0 \\ 0 & 3 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \begin{array}{l} R_1 \rightarrow \frac{1}{2}R_1 \\ R_2 \rightarrow \frac{1}{3}R_2 \end{array} \begin{bmatrix} 1 & \frac{-1}{2} & \frac{1}{2} & | & 0 \\ 0 & 1 & \frac{1}{3} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \begin{array}{l} c_3 = t \\ c_2 = \frac{-1}{3}t \\ c_1 = \frac{-2}{3}t \end{array}$$

If $t=1$, then $(1,1,2) = \frac{2}{3}(2,1,3) + \frac{1}{3}(-1,1,0)$

By the +/- theorem, if $B_w = \{(2,1,3), (-1,1,0)\}$, then $\text{span}(B_w) = \text{span}(B) = W$.

Since B_w is linearly independent, it forms a basis for W .

$$\vec{n} = (2,1,3) \times (-1,1,0) = \begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ -1 & 1 & 0 \end{vmatrix} = (-3, -3, 3)$$

Ergo, W is the plane $x + y - z = 0$.

22. Find all values of t for which S is linearly independent.

a) $S = \{(2,3,5), (-1,t,-1), (-1,-1,t)\}$

$$c_1(2,3,5) + c_2(-1,t,-1) + c_3(-1,-1,t) = (0,0,0)$$

$$\begin{bmatrix} 2 & -1 & -1 & | & 0 \\ 3 & t & -1 & | & 0 \\ 5 & -1 & t & | & 0 \end{bmatrix} \begin{array}{l} R_2 \rightarrow 2R_2 - 3R_1 \\ R_3 \rightarrow 2R_3 - 5R_1 \end{array} \begin{bmatrix} 2 & -1 & -1 & | & 0 \\ 0 & 2t+3 & 1 & | & 0 \\ 0 & 3 & 2t+5 & | & 0 \end{bmatrix}$$

$$\underline{R_2 \leftrightarrow R_3} \begin{bmatrix} 2 & -1 & -1 & | & 0 \\ 0 & 3 & 2t+5 & | & 0 \\ 0 & 2t+3 & 1 & | & 0 \end{bmatrix}$$

$$\underline{R_3 \rightarrow 3R_3 - (2t+3)R_2} \begin{bmatrix} 2 & -1 & -1 & | & 0 \\ 0 & 3 & 2t+5 & | & 0 \\ 0 & 0 & -4t^2 - 16t - 12 & | & 0 \end{bmatrix}$$

$$\begin{array}{l} R_1 \rightarrow \frac{1}{2}R_1 \\ R_1 \rightarrow \frac{1}{3}R_1 \\ R_1 \rightarrow \frac{1}{-4t^2 - 16t - 12}R_1 \end{array} \begin{bmatrix} 1 & \frac{-1}{2} & \frac{-1}{2} & | & 0 \\ 0 & 1 & \frac{2t+5}{3} & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \quad \text{provided } -4t^2 - 16t - 12 \neq 0$$

Thus if $-4t^2 - 16t - 12 \neq 0$ then the solutions is $c_3 = c_2 = c_1 = 0$

and if $-4t^2 - 16t - 12 = 0$ then $c_3 = t$, $c_2 = \frac{-5}{3}t$, $c_1 = \frac{-1}{3}t$

Hence S will be linearly independent if $-4t^2 - 16t - 12 \neq 0$

$$t^2 + 4t + 3 \neq 0$$

$$(t+3)(t+1) \neq 0$$

$$t \neq -3, -1$$

b) $S = \{3x^2 + x + 4, 2x^2 - x, x^2 + tx + 2t\}$

$$c_1(3x^2 + x + 4) + c_2(2x^2 - x) + c_3(x^2 + tx + 2t) = 0$$

$$\begin{aligned} & \begin{bmatrix} 3 & 2 & 1 & | & 0 \\ 1 & -1 & t & | & 0 \\ 4 & 0 & 2t & | & 0 \end{bmatrix} \xrightarrow{\substack{R_2 \rightarrow 3R_2 - R_1 \\ R_3 \rightarrow 3R_3 - 4R_1}} \begin{bmatrix} 3 & 2 & 1 & | & 0 \\ 0 & -5 & 3t-1 & | & 0 \\ 0 & -8 & 6t-4 & | & 0 \end{bmatrix} \\ & \xrightarrow{R_3 \rightarrow 5R_3 - 8R_2} \begin{bmatrix} 3 & 2 & 1 & | & 0 \\ 0 & -5 & 3t-1 & | & 0 \\ 0 & 0 & 6t-12 & | & 0 \end{bmatrix} \\ & \begin{matrix} R_1 \rightarrow \frac{1}{3}R_1 \\ R_2 \rightarrow \frac{-1}{5}R_2 \\ R_3 \rightarrow \frac{-1}{6t-12}R_3 \end{matrix} \begin{bmatrix} 1 & \frac{2}{3} & \frac{1}{3} & | & 0 \\ 0 & 1 & \frac{3t-1}{-5} & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \quad \text{provided } 6t-12 \neq 0 \end{aligned}$$

Thus if $t \neq 2$, then the solutions is $c_3 = c_2 = c_1 = 0$

and if $t = 2$ then $c_3 = t$, $c_2 = \frac{-1}{5}t$, $c_1 = \frac{-1}{5}t$

Hence S will be linearly independent if $t \neq 2$.

23. Do the following sets S span V ?

a) $S = \{(2, -4, 1), (1, 2, -3), (5, -14, 6)\}$ $V = \mathbb{R}^3$

Let $(a, b, c) \in \mathbb{R}^3$.

$$c_1(2, -4, 1) + c_2(1, 2, -3) + c_3(5, -14, 6) = (a, b, c)$$

$$\begin{aligned} & \begin{bmatrix} 2 & 1 & 5 & | & a \\ -4 & 2 & -14 & | & b \\ 1 & -3 & 6 & | & c \end{bmatrix} \xrightarrow{\substack{R_2 \rightarrow R_2 + 2R_1 \\ R_3 \rightarrow 2R_3 - R_1}} \begin{bmatrix} 2 & 1 & 5 & | & a \\ 0 & 4 & -4 & | & b+2a \\ 0 & -7 & 7 & | & 2c-a \end{bmatrix} \\ & \xrightarrow{R_3 \rightarrow 4R_3 + 7R_2} \begin{bmatrix} 2 & 1 & 5 & | & a \\ 0 & 4 & -4 & | & b+2a \\ 0 & 0 & 0 & | & 10a+7b+8c \end{bmatrix} \end{aligned}$$

There is no solution if $10a + 7b + 8c \neq 0$. Thus S does not span \mathbb{R}^3 .

b) $S = \left\{ \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \right\}$ $V = S_{2,2}$ (The set of symmetric 2×2 matrices)

Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in S_{2,2}$

$$\begin{aligned} & c_1 \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} + c_2 \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \\ & \begin{bmatrix} 2 & 3 & 4 & | & a \\ -1 & 2 & 1 & | & b \\ 3 & 1 & 4 & | & c \end{bmatrix} \xrightarrow{\substack{R_2 \rightarrow 2R_2 + R_1 \\ R_3 \rightarrow 2R_3 - 3R_2}} \begin{bmatrix} 2 & 3 & 4 & | & a \\ 0 & 7 & 6 & | & a+2b \\ 0 & -7 & -4 & | & -3a+2c \end{bmatrix} \end{aligned}$$

$$\underline{R_3 \rightarrow R_3 + R_2} \left[\begin{array}{ccc|c} 2 & 3 & 4 & a \\ 0 & 7 & 6 & a+2b \\ 0 & 0 & 2 & -2a+2b+2c \end{array} \right] \begin{array}{l} R_1 \rightarrow \frac{1}{2}R_1 \\ R_2 \rightarrow \frac{1}{7}R_2 \\ R_3 \rightarrow \frac{1}{2}R_3 \end{array} \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & 2 & \frac{1}{2}a \\ 0 & 1 & \frac{6}{7} & \frac{1}{7}a + \frac{2}{7}b \\ 0 & 0 & 1 & -a+b+c \end{array} \right]$$

$$c_3 = -a+b+c, \quad c_2 = a - \frac{4}{7}b - \frac{6}{7}c, \quad c_1 = a - \frac{8}{7}b - \frac{5}{7}c$$

Thus S spans $S_{2,2}$

c) $S = \{x^3 - 2x + 1, x^3 + x^2, x^3 + x^2 + x + 1\}$ $V = P_3$

Since $n(S) = 3 < \dim(P_3) = 4$, then S does not span P_3

24. Are the following sets S bases for the vector space V ?

a) $S = \{(2, -1, 3), (1, 1, 7), (-2, 4, 1)\}$, $V = \mathbb{R}^3$

$$c_1(2, -1, 3) + c_2(1, 1, 7) + c_3(-2, 4, 1) = (0, 0, 0)$$

$$\left[\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ -1 & 1 & 4 & 0 \\ 3 & 7 & 1 & 0 \end{array} \right] \begin{array}{l} R_2 \rightarrow 2R_2 + R_1 \\ R_3 \rightarrow 2R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 11 & 8 & 0 \end{array} \right]$$

$$\underline{R_3 \rightarrow 3R_3 - 11R_2} \left[\begin{array}{ccc|c} 2 & 1 & -2 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & 0 & -42 & 0 \end{array} \right] \begin{array}{l} R_1 \rightarrow \frac{1}{2}R_1 \\ R_2 \rightarrow \frac{1}{3}R_2 \\ R_3 \rightarrow \frac{-1}{42}R_3 \end{array} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$c_1 = c_2 = c_3 = 0$$

Since S is linearly independent and $n(S) = 3 = \dim(\mathbb{R}^3)$, then S is a basis for \mathbb{R}^3

b) $S = \{(-3, 5, 1), (2, -7, 12)\}$, $V = \mathbb{R}^3$

No since $n(S) = 2 \neq \dim(\mathbb{R}^3) = 3$.

c) $S = \{x^2 + 1, x^2 - 1, x^2 + x + 1, x^2 - x - 1\}$, $V = P_2$?

No since $n(S) = 4 \neq \dim(P_2) = 3$

d) $S = \left\{ \begin{bmatrix} 2 & -1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 4 & -1 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 5 & 5 \\ 1 & -2 \end{bmatrix} \right\}$, $V = M_{2,2}$?

$$c_1 \begin{bmatrix} 2 & -1 \\ 2 & 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 & -1 \\ 4 & 3 \end{bmatrix} + c_3 \begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 5 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 2 & 4 & 3 & 5 & 0 \\ -1 & -1 & 3 & 5 & 0 \\ 2 & 4 & -1 & 1 & 0 \\ 3 & 3 & 1 & -2 & 0 \end{array} \right] \begin{array}{l} R_2 \rightarrow 2R_2 + R_1 \\ R_3 \rightarrow R_3 - R_1 \\ R_4 \rightarrow 2R_4 - 3R_1 \end{array} \left[\begin{array}{cccc|c} 2 & 4 & 3 & 5 & 0 \\ 0 & 2 & 9 & 15 & 0 \\ 0 & 0 & -4 & -4 & 0 \\ 0 & -6 & -7 & -19 & 0 \end{array} \right]$$

$$\begin{array}{l}
 \underline{R_4 \rightarrow R_4 + 3R_2} \left[\begin{array}{cccc|c} 2 & 4 & 3 & 5 & 0 \\ 0 & 2 & 9 & 15 & 0 \\ 0 & 0 & -4 & -4 & 0 \\ 0 & 0 & 20 & 26 & 0 \end{array} \right] \xrightarrow{R_4 \rightarrow R_4 + 5R_3} \left[\begin{array}{cccc|c} 2 & 4 & 3 & 5 & 0 \\ 0 & 2 & 9 & 15 & 0 \\ 0 & 0 & -4 & -4 & 0 \\ 0 & 0 & 0 & 6 & 0 \end{array} \right] \\
 \\
 \begin{array}{l}
 R_1 \rightarrow \frac{1}{2}R_1 \\
 R_2 \rightarrow \frac{1}{2}R_2 \\
 R_3 \rightarrow \frac{-1}{4}R_3 \\
 R_4 \rightarrow \frac{1}{6}R_4
 \end{array} \left[\begin{array}{cccc|c} 1 & 2 & \frac{3}{2} & \frac{5}{2} & 0 \\ 0 & 1 & \frac{9}{2} & \frac{15}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]
 \end{array}$$

$$c_4 = c_3 = c_2 = c_1 = 0$$

Since S is linearly independent and $n(S) = 4 = \dim(M_{2,2})$, then S is a basis for $M_{2,2}$

25. Find a basis for $\text{Span}(S)$ if

a) $S = \{(1,2,3,4), (4,3,2,1), (1,1,1,1)\}$

$$\begin{array}{l}
 \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{array} \right] \xrightarrow{\substack{R_2 \rightarrow R_2 - 4R_1 \\ R_3 \rightarrow R_3 - R_1}} \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & -5 & -10 & -15 \\ 0 & -1 & -2 & -3 \end{array} \right] \\
 \\
 \xrightarrow{R_3 \rightarrow 5R_3 - R_1} \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & -5 & -10 & -15 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R_2 \rightarrow \frac{-1}{5}R_2} \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]
 \end{array}$$

Thus a basis for $\text{span}(S)$ is $B_S = \{(1,2,3,4), (0,1,2,3)\}$

Note: If you used the +/- theorem, then you obtain $B_S = \{(1,2,3,4), (4,3,2,1)\}$

b) $S = \{x^2 + 1, x^2 - 1, x^2 + x + 1, x^2 - x - 1\}$

$$c_1(x^2 + 1) + c_2(x^2 - 1) + c_3(x^2 + x + 1) + c_4(x^2 - x - 1) = 0$$

$$\begin{array}{l}
 \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 1 & -1 & 1 & -1 & 0 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 - R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 \end{array} \right] \\
 \\
 \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & 0 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array} \right] \xrightarrow{R_2 \rightarrow \frac{-1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array} \right]
 \end{array}$$

$$c_4 = t, \quad c_3 = t, \quad c_2 = -t, \quad c_1 = -t$$

$$\text{If } t = 1, \text{ then } (x^2 - x - 1) = (x^2 + 1) + (x^2 - 1) - (x^2 + x + 1)$$

By the +/- theorem, if $B_S = \{(x^2 + 1), (x^2 - 1), (x^2 + x + 1)\}$, then

$$\text{span}(S) = \text{span}(B_S).$$

$$c_1(x^2 + 1) + c_2(x^2 - 1) + c_3(x^2 + x + 1) = 0$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{array} \right] \rightarrow \dots \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \quad (\text{from previous solution})$$

Since B_S is linearly independent, then it is a basis for $\text{span}(S)$.

$$\begin{aligned}
 \text{c) } S &= \left\{ \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 3 & 3 \\ 3 & 4 \end{bmatrix} \right\} \\
 c_1 \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} + c_3 \begin{bmatrix} 3 & 3 \\ 3 & 4 \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
 \left[\begin{array}{ccc|c} 2 & -2 & 3 & 0 \\ -1 & 1 & 3 & 0 \\ -1 & 1 & 3 & 0 \\ 1 & -1 & 4 & 0 \end{array} \right] & \begin{array}{l} R_2 \rightarrow 2R_2 + R_1 \\ R_3 \rightarrow 2R_3 + R_1 \\ R_4 \rightarrow 2R_4 - R_1 \end{array} \left[\begin{array}{ccc|c} 2 & -2 & 3 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 5 & 0 \end{array} \right] \begin{array}{l} R_3 \rightarrow R_3 - 2R_2 \\ R_4 \rightarrow 9R_4 - 5R_2 \end{array} \left[\begin{array}{ccc|c} 2 & -2 & 3 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \\
 R_1 \rightarrow \frac{1}{2}R_1 & \begin{bmatrix} 1 & -1 & \frac{3}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & c_3 = 0 \\
 R_2 \rightarrow \frac{1}{9}R_2 & \begin{bmatrix} 1 & -1 & \frac{3}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & c_2 = t \\
 & & c_1 = t
 \end{aligned}$$

$$\text{If } t=1, \text{ then } \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} = - \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

By the +/- theorem, if $B_S = \left\{ \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 3 \\ 3 & 4 \end{bmatrix} \right\}$, then $\text{span}(S) = \text{span}(B_S)$.

Since B_S is linearly independent (the two matrices are not multiples of each other), then it is a basis for $\text{span}(S)$.

26. Find a basis for the vector space W and give the dimension.

$$\text{a) } W = \{(t-s, t+2s, -3t) : t, s \in \mathbb{R}\}$$

$$(t-s, t+2s, -3t) = t(1, 1, 0) + s(-1, 2, 0)$$

If $B = \{(1, 1, 0), (-1, 2, 0)\}$ then $W = \text{span}(B)$. Since B is linearly independent (the two vectors are non-parallel) then B is a basis for W and $\dim(W) = 2$.

$$\text{b) } W = \{\vec{u} \in \mathbb{R}^3 : \vec{u} \perp (5, 4, 3)\}$$

$$\begin{aligned}
 \text{Let } \vec{u} = (x, y, z) \in W. \text{ Then } \vec{u} \bullet (5, 4, 3) &= 0 & z &= t \\
 5x + 4y + 3z &= 0 & y &= s \\
 & & x &= \frac{-3}{5}t - \frac{4}{5}s
 \end{aligned}$$

$$\text{Hence } \vec{u} = \left(\frac{-3}{5}t - \frac{4}{5}s, s, t \right) = \frac{-1}{5}t(3, 0, -5) + \frac{-1}{5}s(4, -5, 0)$$

If $B = \{(3, 0, -5), (4, -5, 0)\}$ then $W = \text{span}(B)$. Since B is linearly independent (the two vectors are non-parallel) then B is a basis for W and $\dim(W) = 2$.

$$c) W = \left\{ \begin{bmatrix} r+2s+3t & r-s \\ r-s & r+3s+4t \end{bmatrix} : t, s \in \mathbb{R} \right\}$$

$$\begin{bmatrix} r+2s+3t & r-s \\ r-s & r+3s+4t \end{bmatrix} = r \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + s \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} + t \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\text{If } B = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \right\} \text{ then } W = \text{span}(B).$$

$$c_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} + c_3 \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 3 & 4 & 0 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \\ R_4 \rightarrow R_4 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right]$$

$$\begin{array}{l} R_3 \rightarrow R_3 - R_2 \\ R_4 \rightarrow 3R_4 + R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R_2 \rightarrow \frac{-1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} c_3 = t \\ c_2 = -t \\ c_1 = -t \end{array}$$

$$\text{If } t=1, \text{ then } \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}, \text{ hence by the +/- theorem, if}$$

$$B_w = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \right\}, \text{ then } W = \text{span}(B) = \text{span}(B_w). \text{ Since } B_w \text{ is linearly}$$

independent (the two matrices are not multiples of each other) then B_w is a basis for W .

$$d) W = \{p(x) \in P_2 : p(3) = 0\}$$

$$\text{Let } p(x) = a + bx + cx^2 \in W.$$

$$\text{Then } p(3) = a + 3b + 9c = 0. \text{ Thus } c = t$$

$$b = s$$

$$a = -3s - 9t$$

$$\text{Thus } p(x) = -3s - 9t + sx + tx^2 = t(x^2 - 9) + s(x - 3)$$

If $B = \{x^2 - 9, x - 3\}$ then $W = \text{span}(B)$. Since B is linearly independent (the two polynomials are not multiples of each other) then B is a basis for W and $\dim(W) = 2$.

$$e) W = \{(2a + 4b, -a - 2b, 3a + 2b + 4c) : a, b, c \in \mathbb{R}\}$$

$$(2a + 4b, -a - 2b, 3a + 2b + 4c) = a(2, -1, 3) + b(4, -2, 2) + c(0, 0, 4)$$

$$\text{If } B = \{(2, -1, 3), (4, -2, 2), (0, 0, 4)\} \text{ then } W = \text{span}(B).$$

$$c_1(2, -1, 3) + c_2(4, -2, 2) + c_3(0, 0, 4) = (0, 0, 0)$$

$$\left[\begin{array}{ccc|c} 2 & 4 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array} \right] \xrightarrow{\substack{R_2 \rightarrow 2R_2 + R_1 \\ R_3 \rightarrow \frac{1}{4}R_3}} \left[\begin{array}{ccc|c} 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{\substack{R_1 \rightarrow \frac{1}{2}R_1 \\ R_2 \leftrightarrow R_3}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$c_3 = 0$$

$$c_2 = t$$

$$c_1 = -2t$$

If $t = 1$, then $(4, -2, 2) = 2(2, -1, 1) + 0(0, 0, 4)$ hence by the +/- theorem, if

$B_W = \{(2, -1, 1), (0, 0, 4)\}$, then $W = \text{span}(B) = \text{span}(B_W)$. Since B_W is linearly independent (the two vectors are nonparallel) then B_W is a basis for W .

$$\text{f) } W = \{(2a + 2c, a + 3b, b + c) : a, b, c \in \mathbb{R}\}$$

$$(2a + 2c, a + 3b, b + c) = a(2, 1, 0) + b(0, 3, 1) + c(2, 0, 1)$$

If $B = \{(2, 1, 0), (0, 3, 1), (2, 0, 1)\}$ then $W = \text{span}(B)$.

$$c_1(2, 1, 0) + c_2(0, 3, 1) + c_3(2, 0, 1) = (0, 0, 0)$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right] \xrightarrow{R_2 \rightarrow 2R_2 - R_1} \left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 6 & -2 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right]$$

$$\xrightarrow{R_3 \rightarrow 6R_3 - R_2} \left[\begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ 0 & 6 & -2 & 0 \\ 0 & 0 & 8 & 0 \end{array} \right] \xrightarrow{\substack{R_1 \rightarrow \frac{1}{2}R_1 \\ R_2 \rightarrow \frac{1}{6}R_2 \\ R_3 \rightarrow \frac{1}{8}R_3}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \quad \begin{array}{l} c_3 = 0 \\ c_2 = 0 \\ c_1 = 0 \end{array}$$

Since B is linearly independent then B is a basis for W and $\dim(W) = 3$.

$$\text{g) } W = \{\vec{u} \in \mathbb{R}^3 : \vec{u} \perp (2, -1, 3) \text{ and } \vec{u} \perp (0, 2, 1)\}$$

$$\text{Let } \vec{u} = (x, y, z) \in W. \text{ Then } \vec{u} \bullet (2, -1, 3) = 0 \quad 2x - y + 3z = 0$$

$$\vec{u} \bullet (0, 2, 1) = 0 \quad 2y + z = 0$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 0 & 2 & 1 & 0 \end{array} \right] \xrightarrow{\substack{R_1 \rightarrow \frac{1}{2}R_1 \\ R_2 \rightarrow \frac{1}{2}R_2}} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \end{array} \right] \quad \begin{array}{l} z = t \\ y = \frac{-1}{2}t \\ x = \frac{-7}{4}t \end{array}$$

$$\text{Hence } \vec{u} = \left(\frac{-7}{4}t, \frac{-1}{2}t, t\right) = \frac{-1}{4}t(7, 2, -4)$$

If $B = \{(7, 2, -4)\}$ then $W = \text{span}(B)$. Since B is linearly independent then B is a basis for W and $\dim(W) = 1$.

27. Find the coordinate vector for \vec{w} in the vector space V relative to the basis S .

$$\text{a) } \vec{w} = (-5, -6, 24), \quad V = \mathbb{R}^3 \text{ and } S = \{(2, -1, 4), (1, 2, 4), (-3, -3, 4)\}.$$

$$c_1(2, -1, 4) + c_2(1, 2, 4) + c_3(-3, -3, 4) = (-5, -6, 24)$$

$$\begin{aligned} & \left[\begin{array}{ccc|c} 2 & 1 & -3 & -5 \\ -1 & 2 & -3 & -6 \\ 4 & 4 & 4 & 24 \end{array} \right] \begin{array}{l} R_2 \rightarrow 2R_2 + R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{array} \left[\begin{array}{ccc|c} 2 & 1 & -3 & -5 \\ 0 & 5 & -9 & -17 \\ 0 & 2 & 10 & 34 \end{array} \right] \\ & \underline{R_3 \rightarrow 5R_3 - 2R_2} \left[\begin{array}{ccc|c} 2 & 1 & -3 & -5 \\ 0 & 5 & -9 & -17 \\ 0 & 0 & 68 & 204 \end{array} \right] \begin{array}{l} R_1 \rightarrow \frac{1}{2}R_1 \\ R_2 \rightarrow \frac{1}{5}R_2 \\ R_3 \rightarrow \frac{1}{68}R_3 \end{array} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \\ 0 & 1 & -\frac{9}{5} & -\frac{17}{5} \\ 0 & 0 & 1 & 3 \end{array} \right] \end{aligned}$$

$$c_3 = 3, c_2 = 2, c_1 = 1$$

$$\vec{w}_S = (1, 2, 3)$$

b) $\vec{w} = (12, -7, 10)$, $V = \mathbb{R}^3$ and $S = \{(3, -1, 5), (1, 2, 3), (2, -1, -1)\}$.

$$c_1(3, -1, 5) + c_2(1, 2, 3) + c_3(2, -1, -1) = (12, -7, 10)$$

$$\begin{aligned} & \left[\begin{array}{ccc|c} 3 & 1 & 2 & 12 \\ -1 & 2 & -1 & -7 \\ 5 & 3 & -1 & 10 \end{array} \right] \begin{array}{l} R_2 \rightarrow 3R_2 + R_1 \\ R_3 \rightarrow 3R_3 - 5R_1 \end{array} \left[\begin{array}{ccc|c} 3 & 1 & 2 & 12 \\ 0 & 7 & -1 & -9 \\ 0 & 4 & -13 & -30 \end{array} \right] \\ & \underline{R_3 \rightarrow 7R_3 - 4R_1} \left[\begin{array}{ccc|c} 3 & 1 & 2 & 12 \\ 0 & 7 & -1 & -9 \\ 0 & 0 & -87 & -174 \end{array} \right] \begin{array}{l} R_1 \rightarrow \frac{1}{3}R_1 \\ R_2 \rightarrow \frac{1}{7}R_2 \\ R_3 \rightarrow \frac{-1}{87}R_3 \end{array} \left[\begin{array}{ccc|c} 1 & \frac{1}{3} & \frac{2}{3} & 4 \\ 0 & 1 & -\frac{1}{7} & -\frac{9}{7} \\ 0 & 0 & 1 & 2 \end{array} \right] \end{aligned}$$

$$c_3 = 2, c_2 = -1, c_1 = 3$$

$$\vec{w}_S = (3, -1, 2)$$

c) $\vec{w} = (4, -3, 5, 3)$, $V = \text{Span}(\{(3, -1, 4, 0), (0, 1, -2, 4), (2, 2, 1, 1)\})$ and $S = \{(3, -1, 4, 0), (0, 1, -2, 4), (2, 2, 1, 1)\}$.

$$c_1(3, -1, 4, 0) + c_2(0, 1, -2, 4) + c_3(2, 2, 1, 1) = (4, -3, 5, 3)$$

$$\begin{aligned} & \left[\begin{array}{cccc|c} 3 & 0 & 2 & 4 \\ -1 & 1 & 2 & -3 \\ 4 & -2 & 1 & 5 \\ 0 & 4 & 1 & 3 \end{array} \right] \begin{array}{l} R_2 \rightarrow 3R_2 + R_1 \\ R_3 \rightarrow 3R_3 - 4R_1 \end{array} \left[\begin{array}{cccc|c} 3 & 0 & 2 & 4 \\ 0 & 3 & 8 & -5 \\ 0 & -6 & -5 & -1 \\ 0 & 4 & 1 & 3 \end{array} \right] \\ & \underline{R_3 \rightarrow R_3 + 2R_2} \quad \underline{R_4 \rightarrow 3R_4 - 4R_2} \left[\begin{array}{cccc|c} 3 & 0 & 2 & 4 \\ 0 & 3 & 8 & -5 \\ 0 & 0 & 11 & -11 \\ 0 & 0 & -29 & 29 \end{array} \right] \quad \underline{R_4 \rightarrow 11R_4 + 29R_3} \left[\begin{array}{cccc|c} 3 & 0 & 2 & 4 \\ 0 & 3 & 8 & -5 \\ 0 & 0 & 11 & -11 \\ 0 & 0 & 0 & 0 \end{array} \right] \\ & \begin{array}{l} R_1 \rightarrow \frac{1}{3}R_1 \\ R_2 \rightarrow \frac{1}{3}R_2 \\ R_3 \rightarrow \frac{1}{11}R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & \frac{4}{3} \\ 0 & 1 & \frac{8}{3} & -\frac{5}{3} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} c_3 = -1 \\ c_2 = 1 \\ c_1 = 2 \end{array} \end{aligned}$$

$$\vec{w}_S = (2, 1, -1)$$

d) $\vec{w} = 9x^2 - x + 11$, $V = P_2$ and $S = \{x^2 + x, x^2 - 1, 2x^2 - 3x + 4\}$.

$$c_1(x^2 + x) + c_2(x^2 - 1) + c_3(2x^2 - 3x + 4) = 9x^2 - x + 11$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 1 & 0 & -3 & -1 \\ 0 & -1 & 4 & 11 \end{array} \right] \xrightarrow{R_2 \rightarrow R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & -1 & -5 & -10 \\ 0 & -1 & 4 & 11 \end{array} \right]$$

$$\xrightarrow{R_3 \rightarrow R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & -1 & -5 & -10 \\ 0 & 0 & 9 & 21 \end{array} \right] \xrightarrow{\begin{array}{l} R_2 \rightarrow -R_2 \\ R_3 \rightarrow \frac{1}{9}R_3 \end{array}} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & 5 & 10 \\ 0 & 0 & 1 & \frac{7}{3} \end{array} \right] \quad \begin{array}{l} c_3 = \frac{7}{3} \\ c_2 = \frac{-5}{3} \\ c_1 = 6 \end{array}$$

$$\vec{w}_S = \left(6, \frac{-5}{3}, \frac{7}{3}\right)$$

28. For each of the given sets A and B in \mathbb{R}^3 ,

i) Find a basis for $\text{span}(A)$ and give a geometrical description.

ii) Find a basis for $\text{span}(B)$ and give a geometrical description.

iii) Find a basis for $\text{span}(A) + \text{span}(B)$ and give a geometrical description.

iv) Find a basis for $\text{span}(A) \cap \text{span}(B)$ and give a geometrical description.

a) $A = \{(2, -1, 3), (4, -2, 6), (1, 5, 1)\}$ $B = \{(1, 2, -1), (3, 1, 1), (0, 5, -4)\}$

(i) $c_1(2, -1, 3) + c_2(4, -2, 6) + c_3(1, 5, 1) = (0, 0, 0)$

$$\left[\begin{array}{ccc|c} 2 & 4 & 1 & 0 \\ -1 & -2 & 5 & 0 \\ 3 & 6 & 1 & 0 \end{array} \right] \xrightarrow{\begin{array}{l} R_2 \rightarrow 2R_2 + R_1 \\ R_3 \rightarrow 2R_3 - 3R_1 \end{array}} \left[\begin{array}{ccc|c} 2 & 4 & 1 & 0 \\ 0 & 0 & 11 & 0 \\ 0 & 0 & -1 & 0 \end{array} \right]$$

$$\xrightarrow{R_3 \rightarrow 11R_3 + R_1} \left[\begin{array}{ccc|c} 2 & 4 & 1 & 0 \\ 0 & 0 & 11 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\begin{array}{l} R_1 \rightarrow \frac{1}{2}R_1 \\ R_2 \rightarrow \frac{1}{11}R_2 \end{array}} \left[\begin{array}{ccc|c} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} c_3 = 0 \\ c_2 = t \\ c_1 = -2t \end{array}$$

If $t = 1$, then $(4, -2, 6) = 2(2, -1, 3) + 0(1, 5, 1)$. By the +/- theorem, if

$B_{\text{span}(A)} = \{(2, -1, 3), (1, 5, 1)\}$, then $\text{span}(A) = \text{span}(B_{\text{span}(A)})$. Since $B_{\text{span}(A)}$ is linearly independent, then $B_{\text{span}(A)}$ is a basis for $\text{span}(A)$.

$$\vec{n} = \begin{vmatrix} i & j & k \\ 2 & -1 & 3 \\ 1 & 5 & 1 \end{vmatrix} = (-16, 1, 11).$$

Geometrically, $\text{span}(A)$ is the plane $16x - y - 11z = 0$.

(ii) $c_1(1, 2, -1) + c_2(3, 1, 1) + c_3(0, 5, -4) = (0, 0, 0)$

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 2 & 1 & 5 & 0 \\ -1 & 1 & -4 & 0 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 + R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 4 & -4 & 0 \end{array} \right] \\ & \xrightarrow{R_3 \rightarrow 5R_3 + 4R_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R_2 \rightarrow -\frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} c_3 = t \\ c_2 = t \\ c_1 = -3t \end{array} \end{aligned}$$

If $t = 1$, then $(0, 5, -4) = 3(1, 2, -1) - (3, 1, 1)$. By the +/- theorem, if

$B_{\text{span}(B)} = \{(1, 2, -1), (3, 1, 1)\}$, then $\text{span}(B) = \text{span}(B_{\text{span}(B)})$. Since $B_{\text{span}(B)}$ is linearly independent, then $B_{\text{span}(B)}$ is a basis for $\text{span}(B)$.

$$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 3 & 1 & 1 \end{vmatrix} = (3, -4, -5).$$

Geometrically, $\text{span}(B)$ is the plane $3x - 4y - 5z = 0$.

(iii) Let $S = \{(2, -1, 3), (1, 5, 1), (1, 2, -1), (3, 1, 1)\}$.

Then $\text{span}(A) + \text{span}(B) = \text{span}(S)$

$$c_1(2, -1, 3) + c_2(1, 5, 1) + c_3(1, 2, -1) + c_4(3, 1, 1) = (0, 0, 0)$$

$$\begin{aligned} & \left[\begin{array}{cccc|c} 2 & 1 & 1 & 3 & 0 \\ -1 & 5 & 2 & 1 & 0 \\ 3 & 1 & -1 & 1 & 0 \end{array} \right] \begin{array}{l} R_2 \rightarrow 2R_2 + R_1 \\ R_3 \rightarrow 2R_3 - 3R_1 \end{array} \left[\begin{array}{cccc|c} 2 & 1 & 1 & 3 & 0 \\ 0 & 11 & 5 & 5 & 0 \\ 0 & -1 & -5 & -7 & 0 \end{array} \right] \\ & \xrightarrow{R_3 \rightarrow 11R_3 + R_2} \left[\begin{array}{cccc|c} 2 & 1 & 1 & 3 & 0 \\ 0 & 11 & 5 & 5 & 0 \\ 0 & 0 & -50 & -72 & 0 \end{array} \right] \end{aligned}$$

$$\begin{aligned} & \begin{array}{l} R_1 \rightarrow \frac{1}{2}R_1 \\ R_2 \rightarrow \frac{1}{11}R_2 \\ R_3 \rightarrow -\frac{1}{50}R_3 \end{array} \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & \frac{5}{11} & \frac{5}{11} & 0 \\ 0 & 0 & 1 & \frac{26}{25} & 0 \end{array} \right] \begin{array}{l} c_4 = t \\ c_3 = \frac{-36}{25}t \\ c_2 = \frac{1}{5}t \\ c_1 = \frac{-22}{25}t \end{array} \end{aligned}$$

If $t = 1$, then $(3, 1, 1) = \frac{22}{25}(2, -1, 3) - \frac{1}{5}(1, 5, 1) + \frac{36}{25}(1, 2, -1)$. By the +/- theorem, if

$B_{\text{span}(A)+\text{span}(B)} = \{(2, -1, 3), (1, 5, 1), (1, 2, -1)\}$, then

$\text{span}(A) + \text{span}(B) = \text{span}(S) = \text{span}(B_{\text{span}(A)+\text{span}(B)})$. Since $B_{\text{span}(A)+\text{span}(B)}$ is linearly independent, then $B_{\text{span}(A)+\text{span}(B)}$ is a basis for $\text{span}(A) + \text{span}(B)$

Geometrically, $\text{span}(A) + \text{span}(B)$ is \mathbb{R}^3 .

(iv) Let $(x, y, z) \in \text{span}(A) \cap \text{span}(B)$.

$$\text{Then } (x, y, z) = c_1(2, -1, 3) + c_2(1, 5, 1)$$

$$(x, y, z) = c_3(1, 2, -1) + c_4(3, 1, 1)$$

$$\text{Hence } c_1(2, -1, 3) + c_2(1, 5, 1) = c_3(1, 2, -1) + c_4(3, 1, 1)$$

$$c_1(2, -1, 3) + c_2(1, 5, 1) - c_3(1, 2, -1) - c_4(3, 1, 1) = (0, 0, 0)$$

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & -3 & 0 \\ -1 & 5 & -2 & -1 & 0 \\ 3 & 1 & 1 & -1 & 0 \end{array} \right] \begin{array}{l} R_2 \rightarrow 2R_2 + R_1 \\ R_3 \rightarrow 2R_3 - 3R_1 \end{array} \left[\begin{array}{cccc|c} 2 & 1 & -1 & -3 & 0 \\ 0 & 11 & -5 & -5 & 0 \\ 0 & -1 & 5 & 7 & 0 \end{array} \right]$$

$$\underline{R_3 \rightarrow 11R_3 + R_2} \left[\begin{array}{cccc|c} 2 & 1 & -1 & -3 & 0 \\ 0 & 11 & -5 & -5 & 0 \\ 0 & 0 & 50 & 72 & 0 \end{array} \right]$$

$$\begin{array}{l} R_1 \rightarrow \frac{1}{2}R_1 \\ R_2 \rightarrow \frac{1}{11}R_2 \\ R_3 \rightarrow \frac{1}{50}R_3 \end{array} \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} & 0 \\ 0 & 1 & -\frac{5}{11} & -\frac{5}{11} & 0 \\ 0 & 0 & 1 & \frac{26}{25} & 0 \end{array} \right] \begin{array}{l} c_4 = t \\ c_3 = \frac{-36}{25}t \\ c_2 = \frac{-1}{5}t \\ c_1 = \frac{22}{25}t \end{array}$$

$$\text{Thus } (x, y, z) = \frac{22}{25}t(2, -1, 3) - \frac{1}{5}t(1, 5, 1) = \frac{1}{25}t(39, -47, 61)$$

If $B_{\text{span}(A) \cap \text{span}(B)} = \{(39, -47, 61)\}$, then $\text{span}(A) \cap \text{span}(B) = \text{span}(B_{\text{span}(A) \cap \text{span}(B)})$.

Since $B_{\text{span}(A) \cap \text{span}(B)}$ is linearly independent, then $B_{\text{span}(A) \cap \text{span}(B)}$ is a basis for $\text{span}(A) \cap \text{span}(B)$.

Geometrically, $\text{span}(A) \cap \text{span}(B)$ is the line $\frac{x}{39} = \frac{y}{-47} = \frac{z}{61}$.

$$\text{b) } A = \{(1, 2, -1), (2, 4, -2), (-1, -2, 1)\} \quad B = \{(3, 1, 2), (2, 9, -7), (1, -8, 9)\}$$

$$\text{(i) Since } (-1, -2, 1) = -(1, 2, -1)$$

$$(2, 4, -2) = 2(1, 2, -1)$$

Then by the +/- theorem, if $B_{\text{span}(A)} = \{(1, 2, -1)\}$, then $\text{span}(A) = \text{span}(B_{\text{span}(A)})$.

Since $B_{\text{span}(A)}$ is linearly independent, then $B_{\text{span}(A)}$ is a basis for $\text{span}(A)$.

Geometrically, $\text{span}(A)$ is the line $x = \frac{1}{2}y = -z$.

$$\text{(ii) } c_1(3, 1, 2) + c_2(2, 9, -7) + c_3(1, -8, 9) = (0, 0, 0)$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 1 & 9 & -8 & 0 \\ 2 & -7 & 9 & 0 \end{array} \right] \begin{array}{l} R_2 \rightarrow 3R_2 - R_1 \\ R_3 \rightarrow 3R_3 - 2R_1 \end{array} \left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 0 & 25 & -25 & 0 \\ 0 & -25 & 25 & 0 \end{array} \right]$$

$$\underline{R_3 \rightarrow R_3 + R_2} \left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 0 & 25 & -25 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} R_1 \rightarrow \frac{1}{3}R_1 \\ R_2 \rightarrow \frac{1}{25}R_2 \end{array} \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} c_3 = t \\ c_2 = t \\ c_1 = -t \end{array}$$

If $t = 1$, then $(1, -8, 9) = (3, 1, 2) - (2, 9, -7)$. By the +/- theorem, if

$B_{\text{span}(B)} = \{(3, 1, 2), (2, 9, -7)\}$, then $\text{span}(B) = \text{span}(B_{\text{span}(B)})$. Since $B_{\text{span}(B)}$ is linearly independent, then $B_{\text{span}(B)}$ is a basis for $\text{span}(B)$.

$$\vec{n} = \begin{vmatrix} i & j & k \\ 3 & 1 & 2 \\ 2 & 9 & -7 \end{vmatrix} = (-25, 25, 25) = -25(1, -1, -1).$$

Geometrically, $\text{span}(B)$ is the plane $x - y - z = 0$.

(iii) Let $S = \{(1, 2, -1), (3, 1, 2), (2, 9, -7)\}$.

Then $\text{span}(A) + \text{span}(B) = \text{span}(S)$

$$c_1(1, 2, -1) + c_2(3, 1, 2) + c_3(2, 9, -7) = (0, 0, 0)$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 2 & 1 & 9 & 0 \\ -1 & 2 & -7 & 0 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 + R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 5 & -5 & 0 \end{array} \right]$$

$$\begin{array}{l} \underline{R_3 \rightarrow R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \underline{R_2 \rightarrow \frac{-1}{5}R_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} c_3 = t \\ c_2 = t \\ c_1 = -5t \end{array} \end{array}$$

If $t = 1$, then $(2, 9, -7) = 5(1, 2, -1) - (3, 1, 2)$. By the +/- theorem, if

$B_{\text{span}(A) + \text{span}(B)} = \{(1, 2, -1), (3, 1, 2)\}$, then

$\text{span}(A) + \text{span}(B) = \text{span}(S) = \text{span}(B_{\text{span}(A) + \text{span}(B)})$. Since $B_{\text{span}(A) + \text{span}(B)}$ is linearly independent (two nonparallel vectors), then $B_{\text{span}(A) + \text{span}(B)}$ is a basis for $\text{span}(A) + \text{span}(B)$

$$\vec{n} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 3 & 1 & 2 \end{vmatrix} = (5, -5, -5) = 5(1, -1, -1)$$

Geometrically, $\text{span}(A) + \text{span}(B)$ is the plane $x - y - z = 0$.

(iv) Let $(x, y, z) \in \text{span}(A) \cap \text{span}(B)$.

Then $(x, y, z) = c_1(1, 2, -1)$

$$(x, y, z) = c_2(3, 1, 2) + c_3(2, 9, -7)$$

Hence $c_1(1, 2, -1) - c_2(3, 1, 2) - c_3(2, 9, -7) = (0, 0, 0)$

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 2 & -1 & -9 & 0 \\ -1 & -2 & 7 & 0 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 + R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & -5 & 5 & 0 \end{array} \right]$$

$$\frac{R_3 \rightarrow R_3 + R_2}{\left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]} \quad \frac{R_2 \rightarrow \frac{1}{5}R_2}{\left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]} \quad \begin{array}{l} c_3 = t \\ c_2 = t \\ c_1 = -5t \end{array}$$

$$\text{Thus } (x, y, z) = -5t(1, 2, -1)$$

If $B_{\text{span}(A) \cap \text{span}(B)} = \{(1, 2, -1)\}$, then $\text{span}(A) \cap \text{span}(B) = \text{span}(B_{\text{span}(A) \cap \text{span}(B)})$.

Since $B_{\text{span}(A) \cap \text{span}(B)}$ is linearly independent, then $B_{\text{span}(A) \cap \text{span}(B)}$ is a basis for $\text{span}(A) \cap \text{span}(B)$.

Geometrically, $\text{span}(A) \cap \text{span}(B)$ is the line $x = \frac{1}{2}y = -z$.

29. For the given matrices A , find a basis for the row space, for the column space, for the nullspace, and find the rank and nullity.

$$\text{a) } A = \begin{bmatrix} 0 & -1 & 4 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc} 0 & -1 & 4 \\ 0 & 2 & 1 \end{array} \right] \frac{R_2 \rightarrow R_2 + 2R_1}{\left[\begin{array}{ccc} 0 & -1 & 4 \\ 0 & 0 & 9 \end{array} \right]} \begin{array}{l} R_1 \rightarrow -R_1 \\ R_2 \rightarrow \frac{1}{9}R_2 \end{array} \left[\begin{array}{ccc} 0 & 1 & -4 \\ 0 & 0 & 1 \end{array} \right]$$

$$B_{RS} = \{(0, 1, -4), (0, 0, 1)\} \quad \text{rank}(A) = 2$$

$$B_{CS} = \{(-1, 2), (4, 1)\}$$

Solving $AX = 0$ we obtain $z = 0, y = 0, x = t$,

$$(x, y, z) = t(1, 0, 0)$$

$$B_{SS} = \{(1, 0, 0)\} \quad \text{nullity}(A) = 1$$

$$\text{b) } A = \begin{bmatrix} 1 & -2 & 2 & 3 & 1 \\ 2 & -4 & 4 & 4 & 2 \\ 3 & -6 & 6 & 7 & 3 \\ 5 & -10 & 10 & 11 & 5 \end{bmatrix}$$

$$\left[\begin{array}{ccccc} 1 & -2 & 2 & 3 & 1 \\ 2 & -4 & 4 & 4 & 2 \\ 3 & -6 & 6 & 7 & 3 \\ 5 & -10 & 10 & 11 & 5 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \\ R_4 \rightarrow R_4 - 5R_1 \end{array} \left[\begin{array}{ccccc} 1 & -2 & 2 & 3 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{array} \right]$$

$$\begin{array}{l} R_3 \rightarrow R_3 - R_2 \\ R_4 \rightarrow R_4 - 2R_2 \end{array} \left[\begin{array}{ccccc} 1 & -2 & 2 & 3 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \frac{R_2 \rightarrow \frac{-1}{2}R_2}{\left[\begin{array}{ccccc} 1 & -2 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]}$$

$$B_{RS} = \{(1, -2, 2, 3, 1), (0, 0, 0, 1, 0)\} \quad \text{rank}(A) = 2$$

$$B_{CS} = \{(1, 2, 3, 5), (3, 4, 7, 11)\}$$

Solving $AX = 0$ we obtain $u = t$, $w = 0$, $z = s$, $y = r$, $x = 2r - 2s - t$

$$(x, y, z, w, u) = r(2, 1, 0, 0, 0) + s(-2, 0, 1, 0, 0) + t(-1, 0, 0, 0, 1)$$

$$B_{SS} = \{(2, 1, 0, 0, 0), (-2, 0, 1, 0, 0), (-1, 0, 0, 0, 1)\} \quad \text{nullity}(A) = 3$$

30. Find a basis, and the dimension, for the solution space of $AX = 0$.

a) $x - 2y + z - w = 0$

$$3x + y + w = 0$$

$$4x + 6y - 2z + 4w = 0$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 3 & 1 & 0 & 1 & 0 \\ 4 & 6 & -2 & 4 & 0 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - 4R_1 \end{array} \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 7 & -3 & 4 & 0 \\ 0 & 14 & -6 & 8 & 0 \end{array} \right]$$

$$\begin{array}{l} R_3 \rightarrow R_3 - 2R_2 \\ R_2 \rightarrow \frac{1}{7}R_2 \end{array} \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 7 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 1 & \frac{-3}{7} & \frac{4}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$w = t, \quad z = s, \quad y = \frac{3}{7}s - \frac{4}{7}t, \quad x = \frac{-1}{7}s - \frac{1}{7}t$$

$$(x, y, z, t) = \left(\frac{-1}{7}s - \frac{1}{7}t, \frac{3}{7}s - \frac{4}{7}t, s, t \right) = s\left(\frac{-1}{7}, \frac{3}{7}, 1, 0\right) + t\left(\frac{-1}{7}, \frac{-4}{7}, 0, 1\right)$$

$$B_{SS} = \left\{ \left(\frac{-1}{7}, \frac{3}{7}, 1, 0\right), \left(\frac{-1}{7}, \frac{-4}{7}, 0, 1\right) \right\} \quad \dim(SS) = 2$$

b) $x - y + 3z = 0$

$$2x + 5y + 6z = 0$$

$$x - 8y + 3z = 0$$

$$2x + y + 6z = 0$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 2 & 5 & 6 & 0 \\ 1 & -8 & 3 & 0 \\ 2 & 1 & 6 & 0 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - R_1 \\ R_4 \rightarrow R_4 - 2R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array} \right]$$

$$\begin{array}{l} R_3 \rightarrow 8R_3 + 7R_2 \\ R_4 \rightarrow 4R_4 - R_2 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} R_2 \rightarrow \frac{1}{8}R_2 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$z = t, \quad y = 0, \quad x = -3t$$

$$(x, y, z) = t(-3, 0, 1)$$

$$B_{SS} = \{(-3, 0, 1)\} \quad \dim(SS) = 1$$

31. Write the solution to the system $AX = b$ in the form $X = X_p + X_h$, where X_h is a solution of $AX = 0$ and X_p is a particular solution of $AX = b$.

$$\begin{aligned} 2x - 3y + 4z - w &= 6 \\ x + y - 3z + 2w &= -2 \\ x - 4y + 7z - 3w &= 8 \\ 4x - y - 2z + 3w &= 2 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 2 & -3 & 4 & -1 & 6 \\ 1 & 1 & -3 & 2 & -2 \\ 1 & -4 & 7 & -3 & 8 \\ 4 & -1 & -2 & 3 & 2 \end{array} \right] \begin{array}{l} R_2 \rightarrow 2R_2 - R_1 \\ R_3 \rightarrow 2R_3 - R_1 \\ R_4 \rightarrow R_4 - 2R_1 \end{array} \left[\begin{array}{cccc|c} 2 & -3 & 4 & -1 & 6 \\ 0 & 5 & -10 & 5 & -10 \\ 0 & -5 & 10 & -5 & 10 \\ 0 & 5 & -10 & 5 & -10 \end{array} \right]$$

$$\begin{array}{l} R_3 \rightarrow R_3 + R_2 \\ R_4 \rightarrow R_4 - R_2 \end{array} \left[\begin{array}{cccc|c} 2 & -3 & 4 & -1 & 6 \\ 0 & 5 & -10 & 5 & -10 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} R_1 \rightarrow \frac{1}{2}R_1 \\ R_2 \rightarrow \frac{1}{5}R_2 \end{array} \left[\begin{array}{cccc|c} 1 & -\frac{3}{2} & 2 & -\frac{1}{2} & 3 \\ 0 & 1 & -2 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$w = t, \quad z = s, \quad y = 2s - t + 2, \quad x = s - t$$

$$(x, y, z, w) = (0, -2, 0, 0) + s(1, 2, 1, 0) + t(-1, -1, 0, 1)$$

32. Find the standard matrix for $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

- a) T rotates a vector through an angle of 60° , then reflects that vector about the line $y = x$ and then dilates by a factor of 6.

$$[T] = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3\sqrt{3} & 3 \\ 3 & -3\sqrt{3} \end{bmatrix}$$

- b) T projects a vector onto the line $y = 3x$.

The direction vector for the line is $\vec{u} = (1, 3)$.

$$\text{Thus } T(x, y) = \text{proj}_{\vec{u}}(x, y) = \frac{\vec{u} \cdot (x, y)}{\vec{u} \cdot \vec{u}} \vec{u} = \frac{x + 3y}{10} (1, 3) = \left(\frac{1}{10}x + \frac{3}{10}y, \frac{3}{10}x + \frac{9}{10}y \right)$$

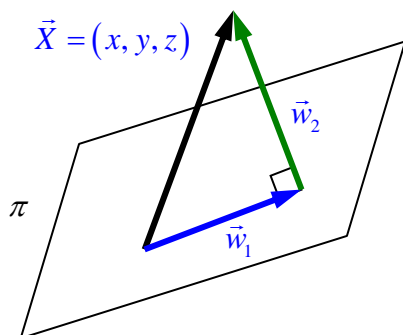
$$\text{Ergo, } [T] = \begin{bmatrix} \frac{1}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{9}{10} \end{bmatrix}$$

33. Find the standard matrix for $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$.

- a) T projects a vector onto the xy -plane.

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

b) T projects a vector onto the plane $x + 2y + 3z = 0$.



$$\vec{n} = (1, 2, 3)$$

$$\vec{w}_2 = \text{proj}_{\vec{n}} \vec{X} = \frac{\vec{X} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n} = \frac{(x, y, z) \cdot (1, 2, 3)}{(1, 2, 3) \cdot (1, 2, 3)} (1, 2, 3) = \frac{x + 2y + 3z}{14} (1, 2, 3)$$

$$= \left(\frac{1}{14}x + \frac{2}{7}y + \frac{3}{14}z, \frac{2}{7}x + \frac{4}{7}y + \frac{9}{14}z, \frac{3}{14}x + \frac{6}{7}y + \frac{9}{14}z \right)$$

$$\vec{w}_1 = \vec{X} - \vec{w}_2$$

$$= (x, y, z) - \left(\frac{1}{14}x + \frac{2}{7}y + \frac{3}{14}z, \frac{2}{7}x + \frac{4}{7}y + \frac{9}{14}z, \frac{3}{14}x + \frac{6}{7}y + \frac{9}{14}z \right)$$

$$= \left(\frac{13}{14}x - \frac{2}{7}y - \frac{3}{14}z, -\frac{2}{7}x + \frac{3}{7}y - \frac{3}{7}z, -\frac{3}{14}x - \frac{3}{7}y + \frac{5}{14}z \right)$$

$$\text{Ergo, } [T] = \begin{bmatrix} \frac{13}{14} & -\frac{2}{7} & -\frac{3}{14} \\ -\frac{2}{7} & \frac{3}{7} & -\frac{3}{7} \\ -\frac{3}{14} & -\frac{3}{7} & \frac{5}{14} \end{bmatrix}$$

34. For each of the following linear transformations T , find

- (i) a basis for the range of T
- (ii) the rank of T
- (iii) a basis for the kernel of T
- (iv) the nullity of T

$$\text{a) } [T] = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 4 & -2 & 1 & 3 \\ 2 & -1 & -2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 3 & 1 \\ 4 & -2 & 1 & 3 \\ 2 & -1 & -2 & 2 \end{bmatrix} \xrightarrow{\substack{R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - R_1}} \begin{bmatrix} 2 & -1 & 3 & 1 \\ 0 & 0 & -5 & 1 \\ 0 & 0 & -5 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \rightarrow R_3 - R_2} \begin{bmatrix} 2 & -1 & 3 & 1 \\ 0 & 0 & -5 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\substack{R_1 \rightarrow \frac{1}{2}R_1 \\ R_2 \rightarrow -\frac{1}{5}R_2}} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If $B_{\text{range}(T)} = \{(2, 4, 2), (3, 1, -2)\}$, then $B_{\text{range}(T)}$ is a basis for the column space of $[T]$, hence is a basis for the range of T . $\text{rank}(T) = 2$.

Solving $[T]X = 0$ gives us $z = t$

$$y = s$$

$$x = \frac{1}{3}s - \frac{2}{3}t$$

Thus $X = \frac{1}{3}s(1, 3, 0) - \frac{1}{3}t(2, 0, -3)$.

Ergo, if $B_{\ker(T)} = \{(1, 3, 0), (2, 0, -3)\}$ then $B_{\ker(T)}$ is a basis for the solution space, thus a basis for the kernel of T . $\text{nullity}(T) = 2$.