MATHEMATICS 201-NYC-05
Vectors and Matrices
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Properties of Vectors

Properties of Vector Sum

Let V be the set of all algebraic vectors having the same dimension and let $\vec{u}, \vec{v}, \vec{w} \in V$.

1. $\vec{u} + \vec{v} \in V$ — closure under addition
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ — commutativity
3. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ — associativity
4. $\vec{0} + \vec{u} = \vec{u}$
5. $\vec{u} + (-\vec{u}) = \vec{0}$

Properties of scalar multiplication

Let V be the set of all algebraic vectors having the same dimension with $\vec{u}, \vec{v} \in V$ and k, l scalars.

6. $k\vec{u} \in V$ — closure under scalar multiplication
7. $(k + l)\vec{u} = k\vec{u} + l\vec{u}$ — distributivity
8. $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$ — distributivity
9. $k(l\vec{u}) = (kl)\vec{u}$ — associativity
10. $1\vec{u} = \vec{u}$

Definition of the dot product

$$\vec{u} \bullet \vec{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

Properties of the dot product

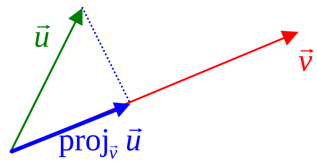
If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors in $\mathbb{R}^n$ and c is a scalar, then we have the following properties.

1. $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$
2. $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$
3. $c(\vec{u} \bullet \vec{v}) = (c\vec{u}) \bullet \vec{v} = \vec{u} \bullet (c\vec{v})$
4. $\vec{u} \bullet \vec{u} = \|\vec{u}\|^2$
5. $\vec{u} \bullet \vec{u} \geq 0$ and $\vec{u} \bullet \vec{u} = 0$ if and only if $\vec{u} = 0$
6. $|\vec{u} \bullet \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$ (*Cauchy Scharwz Inequality*)
7. $\cos\theta = \dfrac{\vec{u} \bullet \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$ (*Defintion of the angle between two vectors*)

Orthogonal Projections

If $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$ such that $\vec{v} \neq 0$, then the orthogonal projection of $\vec{u}$ onto $\vec{v}$ is given by

$$\text{proj}_{\vec{v}}\,\vec{u} = \frac{\vec{u}\bullet\vec{v}}{\vec{v}\bullet\vec{v}}\vec{v}$$



Definition of the cross product

$$\vec{u}\times\vec{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\hat{\mathbf{i}} + (u_3v_1 - u_1v_3)\hat{\mathbf{j}} + (u_1v_2 - u_2v_1)\hat{\mathbf{k}}$$

Properties of the Cross Product

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^3$ and k a scalar.

1. $\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$
2. $k(\vec{u}\times\vec{v}) = (k\vec{u})\times\vec{v} = \vec{u}\times(k\vec{v})$
3. $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$
4. $(\vec{u}+\vec{v})\times\vec{w} = (\vec{u}\times\vec{w}) + (\vec{v}\times\vec{w})$
5. $\vec{u}\times\vec{0} = \vec{0}\times\vec{u} = \vec{0}$
6. $\vec{u}\times\vec{u} = \vec{0}$
7. $\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\theta$

Relationships involving Cross and Dot Products

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^3$.

1. $\vec{u}\bullet(\vec{u}\times\vec{v}) = 0$ $\qquad\qquad \vec{u}\perp\vec{u}\times\vec{v}$
2. $\vec{v}\bullet(\vec{u}\times\vec{v}) = 0$ $\qquad\qquad \vec{v}\perp\vec{u}\times\vec{v}$
3. $\|\vec{u}\times\vec{v}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u}\bullet\vec{v})^2$
4. $\vec{u}\times(\vec{v}\times\vec{w}) = (\vec{u}\bullet\vec{w})\vec{v} - (\vec{u}\bullet\vec{v})\vec{w}$
5. $(\vec{u}\times\vec{v})\times\vec{w} = (\vec{u}\bullet\vec{v})\vec{w} - (\vec{v}\bullet\vec{w})\vec{u}$
6. $\vec{u}\bullet(\vec{v}\times\vec{w}) = \vec{w}\bullet(\vec{u}\times\vec{v}) = \vec{v}\bullet(\vec{w}\times\vec{u})$