MATHEMATICS 201-NYC-05
Vectors and Matrices
Martin Huard
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Properties of Matrices

Properties of Matrix Arithmetic

1. $A+B=B+A$ (Commutative law for addition)
2. $A+(B+C)=(A+B)+C$ (Associative law for addition)
3. $A(BC)=(AB)C$ (Associative law for multiplication)
4. $A(B+C)=AB+AC$ (Left distributive law)
5. $(A+B)C=AC+BC$ (Right distributive law)
6. $a(A+B)=aA+aB$
7. $(a+b)A=aA+bA$
8. $a(bA)=(ab)A$
9. $a(AB)=(aA)B$

Properties of Zero Matrices

1. $A_{m\times n}+0_{m\times n}=0_{m\times n}+A_{m\times n}=A_{m\times n}$
2. $A_{m\times n}-A_{m\times n}=0_{m\times n}$
3. $0_{m\times n}-A_{m\times n}=-A_{m\times n}$
4. $A_{m\times r}0_{r\times n}=0_{m\times n}$
   $0_{m\times r}A_{r\times n}=0_{m\times n}$

Properties of Identity Matrices

1. $A_{m\times n}I_{n\times n}=A_{m\times n}$
2. $I_{m\times m}A_{m\times n}=A_{m\times n}$

Properties of the Transpose

1. $(A^T)^T=A$
2. $(A+B)^T=A^T+B^T$
3. $(kA)^T=kA^T$
4. $(AB)^T=B^TA^T$

Properties of the Trace

1. $tr(A+B)=tr(A)+tr(B)$
2. $tr(kA)=k\cdot tr(A)$
3. $tr(AB)=tr(BA)$

Exponents

Definition: If A is a square matrix then for nonnegative integers n,

$$A^0 = I \qquad A^n = \underbrace{AA\cdots A}_{n \text{ factors}}$$

If A is invertible, then, for negative integers, we have

$$A^{-n} = \left(A^{-1}\right)^n = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{n \text{ factors}}$$

Properties: 1. $A^r A^s = A^{r+s}$

2. $\left(A^r\right)^s = A^{rs}$

Properties of Inverses

If A and B are invertible matrices, then:

1. A^{-1} is invertible with $\left(A^{-1}\right)^{-1} = A$
2. kA is invertible with $\left(kA\right)^{-1} = \frac{1}{k}A^{-1}$
3. A^T is invertible with $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$
4. AB is invertible with $\left(AB\right)^{-1} = B^{-1}A^{-1}$
5. A^n is invertible with $\left(A^n\right)^{-1} = \left(A^{-1}\right)^n$
6. $A^{-1} = \dfrac{1}{\det(A)}\operatorname{adj}(A)$

Matrices as products of Elementary Matrices

1. $A^{-1} = E_k E_{k-1} \cdots E_3 E_2 E_1$
2. $A = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_k^{-1}$

Properties of determinants

1. $\det\left(A^T\right) = \det(A)$
2. $\det(kA) = k^n \det(A)$
3. $\det(AB) = \det(A)\det(B)$
4. $\det\left(A^{-1}\right) = \dfrac{1}{\det(A)}$ (if A is invertible)