MATHEMATICS 201-NYC-05
Vectors and Matrices
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IX - Dot Product

1. Find (i) $\vec{u} \bullet \vec{v}$ (ii) $\vec{u} \bullet \vec{u}$ (iii) $\|\vec{u}\|$ and (iv) $(\vec{u} \bullet \vec{v})\vec{v}$
   a) $\vec{u} = (3,4)$, $\vec{v} = (2,-3)$
   b) $\vec{u} = (1,-2,3)$, $\vec{v} = (4,5-1)$
   c) $\vec{u} = (4,0,-3,5)$, $\vec{v} = (0,2,5,4)$
   d) $\vec{u} = (0,1,-3,4,1)$, $\vec{v} = (-4,5,1,2,2)$

2. Find the angle θ between the given vectors.
   a) $\vec{u} = (3,4)$, $\vec{v} = (-4,3)$
   b) $\vec{u} = (1,-2,3)$, $\vec{v} = (4,5-1)$
   c) $\vec{u} = (4,0,-3,5)$, $\vec{v} = (-8,0,6,-10)$
   d) $\vec{u} = (0,1,-3,4,1)$, $\vec{v} = (-4,5,1,2,2)$

3. Determine all vectors $\vec{w}$ that are perpendicular to $\vec{u}$.
   a) $\vec{u} = (-1,5)$
   b) $\vec{u} = (3,0)$
   c) $\vec{u} = (-1,1,2)$
   d) $\vec{u} = (0,1,0,0)$

4. Find all vectors $\vec{w}$ that are orthogonal to $\vec{u}$ and $\vec{v}$
   a) $\vec{u} = (-1,2,2)$, $\vec{v} = (-1,2,5)$
   b) $\vec{u} = (1,0,-3)$, $\vec{v} = (-1,1,1)$
   c) $\vec{u} = (-1,2,3)$, $\vec{v} = (2,-4,-6)$
   d) $\vec{u} = (-1,3,2,1)$, $\vec{v} = (2,-5,1,-2)$

5. Determine whether $\vec{u}$ and $\vec{v}$ are orthogonal, parallel or neither.
   a) $\vec{u} = (3,0)$, $\vec{v} = (1,1)$
   b) $\vec{u} = \left(\frac{1}{2},-\frac{2}{3}\right)$, $\vec{v} = (4,3)$
   c) $\vec{u} = (4,0,-3)$, $\vec{v} = (0,-2,5)$
   d) $\vec{u} = (1,-3,2)$, $\vec{v} = (-1,-1,-1)$
   e) $\vec{u} = (4,-5,2,6)$, $\vec{v} = \left(-2,\frac{5}{2},-1,-3\right)$

6. Determine whether the sets in $\mathbb{R}^3$ are orthogonal and orthonormal.
   a) $\left\{(0,1,0),\left(\frac{-4}{5},0,\frac{3}{5}\right),\left(\frac{3}{5},0,\frac{4}{5}\right)\right\}$
   b) $\left\{(1,-2,3),(1,2,1),(3,0,-1)\right\}$
   c) $\left\{\left(\frac{1}{5},\frac{1}{5},\frac{1}{5}\right),\left(-\frac{1}{2},\frac{1}{2},0\right),\left(\frac{1}{3},\frac{1}{3},-\frac{2}{3}\right)\right\}$

7. Verify the Cauchy-Schwarz Inequality and the Triangular Inequality for the given vectors.
   a) $\vec{u} = (3,0)$, $\vec{v} = (1,1)$
   b) $\vec{u} = (-3,2)$, $\vec{v} = (4,3)$
   c) $\vec{u} = (2,2,2)$, $\vec{v} = (0,-2,1)$

8. Determine if the triangle ABC is a right triangle, and if so, find where right angle is.
   a) $A(1,0)$, $B(2,3)$ and $C(6,0)$
   b) $A(6,5)$, $B(1,3)$ and $C(3,-2)$
   c) $A(1,2,3)$, $B(-1,-3,2)$ and $C(5,0,5)$

9. Prove that if $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors in $\mathbb{R}^n$, then
$$\vec{u}\bullet(\vec{v}+\vec{w})=\vec{u}\bullet\vec{v}+\vec{u}\bullet\vec{w}$$

10. Prove that if $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then
$$\|\vec{u}+\vec{v}\|^2+\|\vec{u}-\vec{v}\|^2=2\|\vec{u}\|^2+2\|\vec{v}\|^2$$

11. Prove that if $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, then
$$\vec{u}\bullet\vec{v}=\tfrac{1}{4}\|\vec{u}+\vec{v}\|^2-\tfrac{1}{4}\|\vec{u}-\vec{v}\|^2$$

12. Prove that the diagonals of a rectangle are perpendicular if and only if the rectangle is a square.

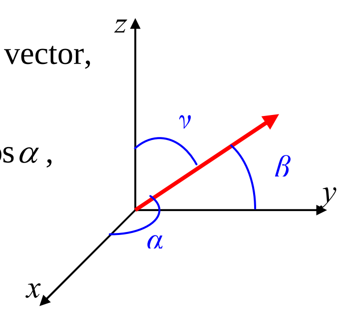
13. Let ABCD be a parallelogram. Prove that
$$\left\|\overrightarrow{AC}\right\|^2+\left\|\overrightarrow{BD}\right\|^2=\left\|\overrightarrow{AB}\right\|^2+\left\|\overrightarrow{BC}\right\|^2+\left\|\overrightarrow{CD}\right\|^2+\left\|\overrightarrow{DA}\right\|^2$$

14. Find the projection of (i) $\vec{u}$ onto $\vec{v}$ (ii) $\vec{v}$ onto $\vec{u}$
   a) $\vec{u}=(3,4)$, $\vec{v}=(2,-3)$
   b) $\vec{u}=(1,-2,3)$, $\vec{v}=(4,5-1)$
   c) $\vec{u}=(4,0,-3,5)$, $\vec{v}=(0,2,5,4)$
   d) $\vec{u}=(0,1,-3,4,1)$, $\vec{v}=(-4,5,1,2,2)$

15. Prove $U\bullet AV=A^TU\bullet V$ where A is an $n\times n$ matrix using the matrix formula for the dot product.

16. Prove that $\vec{u}\bullet(\vec{v}+\vec{w})=\vec{u}\bullet\vec{v}+\vec{u}\bullet\vec{w}$ using the matrix formula for the dot product.

17. Let $\mathbf{\hat{i}}$, $\mathbf{\hat{j}}$ and $\mathbf{\hat{k}}$ be unit vectors along the positive x, y and z axes of a rectangular coordinate system in 3-space. If $\vec{v}=(a,b,c)$ is a nonzero vector, then the angles α, β and γ between $\vec{v}$ and the vectors $\mathbf{\hat{i}}$, $\mathbf{\hat{j}}$ and $\mathbf{\hat{k}}$, respectively, are called the ***direction angles*** of $\vec{v}$, and the numbers $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are called the ***direction cosines*** of $\vec{v}$.



   a) Show that $\cos\alpha=\dfrac{a}{\|\vec{v}\|}$. Find $\cos\beta$ and $\cos\gamma$.
   b) Prove that $(\cos\alpha,\cos\beta,\cos\gamma)$ is a unit vector.
   c) Show that $\vec{v}=\|\vec{v}\|(\cos\alpha,\cos\beta,\cos\gamma)$. This is called the ***polar decomposition*** of $\vec{v}$
   d) Show that two vectors $\vec{v}_1$ and $\vec{v}_2$ in 3-space are perpendicular if their direction cosines satisfy $\cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2=0$.
   e) Find the polar decomposition, the direction cosines, and the direction angles for $\vec{u}=(\sqrt{3},0,-1)$

Answers

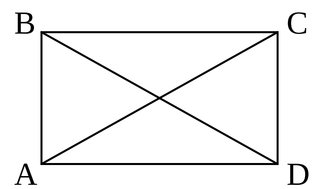
1. a) i. -6 ii. 25 iii. 5 iv. $(-12,18)$
   b) i. -9 ii. 14 iii. $\sqrt{14}$ iv. $(-36,-45,9)$
   c) i. 5 ii. 50 iii. $5\sqrt{2}$ iv. $(0,10,25,20)$
   d) i. 12 ii. 27 iii. $3\sqrt{3}$ iv. $(-48,60,12,24,24)$
2. a) 90° b) 111.8° c) 180° d) 70.9°
3. a) $t(5,1)$ b) $t(0,1)$ c) $s(1,1,0)+t(2,0,1)$ d) $r(1,0,0,0)+s(0,0,1,0)+t(0,0,0,1)$
4. a) $t(2,1,0)$ b) $t(3,2,1)$ c) $s(2,1,0)+t(3,0,1)$ d) $s(1,0,0,1)+t(-13,-5,1,0)$
5. a) neither b) orthogonal c) neither d) orthogonal e) parallel
6. a) Orthogonal and orthonormal b) Neither c) Orthogonal
8. a) No b) Yes in B c) Yes in A
9. Let $\vec{u}=(u_1,u_2,\ldots,u_n)$, $\vec{v}=(v_1,v_2,\ldots,v_n)$ and $\vec{w}=(w_1,w_2,\ldots,w_n)$.

$$\begin{aligned}\vec{u}\bullet(\vec{v}+\vec{w}) &= (u_1,u_2,\ldots,u_n)\bullet(v_1+w_1,v_2+w_2,\ldots,v_n+w_n)\\ &= u_1(v_1+w_1)+u_2(v_2+w_2)+\ldots+u_n(v_n+w_n)\\ &= u_1v_1+u_1w_1+u_2v_2+u_2w_2+\ldots+u_nv_n+u_nw_n\\ &= (u_1v_1+u_2v_2+\ldots+u_nv_n)+(u_1w_1+u_2w_2+\ldots+u_nw_n)\\ &= \vec{u}\bullet\vec{v}+\vec{u}\bullet\vec{w}\end{aligned}$$

10.
$$\begin{aligned}\|\vec{u}+\vec{v}\|^2+\|\vec{u}-\vec{v}\|^2 &= (\vec{u}+\vec{v})\bullet(\vec{u}+\vec{v})+(\vec{u}-\vec{v})\bullet(\vec{u}-\vec{v})\\ &= \vec{u}\bullet\vec{u}+2\vec{u}\bullet\vec{v}+\vec{v}\bullet\vec{v}+\vec{u}\bullet\vec{u}-2\vec{u}\bullet\vec{v}+\vec{v}\bullet\vec{v}\\ &= 2\|\vec{u}\|^2+2\|\vec{v}\|^2\end{aligned}$$

11.
$$\begin{aligned}\tfrac{1}{4}\|\vec{u}+\vec{v}\|^2-\tfrac{1}{4}\|\vec{u}-\vec{v}\|^2 &= \tfrac{1}{4}(\vec{u}+\vec{v})\bullet(\vec{u}+\vec{v})-\tfrac{1}{4}(\vec{u}-\vec{v})\bullet(\vec{u}-\vec{v})\\ &= \tfrac{1}{4}\vec{u}\bullet\vec{u}+\tfrac{1}{2}\vec{u}\bullet\vec{v}+\tfrac{1}{4}\vec{v}\bullet\vec{v}-\tfrac{1}{4}\vec{u}\bullet\vec{u}+\tfrac{1}{2}\vec{u}\bullet\vec{v}-\tfrac{1}{4}\vec{v}\bullet\vec{v}\\ &= \vec{u}\bullet\vec{v}\end{aligned}$$

12. The diagonals are perpendicular if and only if $\overrightarrow{AC}\bullet\overrightarrow{BD}=0$



$$\begin{aligned}\overrightarrow{AC}\bullet\overrightarrow{BD} &= \left(\overrightarrow{AB}+\overrightarrow{BC}\right)\bullet\left(\overrightarrow{BC}+\overrightarrow{CD}\right)\\ &= \overrightarrow{AB}\bullet\overrightarrow{BC}+\overrightarrow{AB}\bullet\overrightarrow{CD}+\overrightarrow{BC}\bullet\overrightarrow{BC}+\overrightarrow{BC}\bullet\overrightarrow{CD}\\ &= 0+\overrightarrow{AB}\bullet\overrightarrow{CD}+\overrightarrow{BC}\bullet\overrightarrow{BC}+0\end{aligned}$$

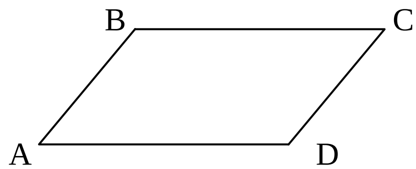
Since ABCD is a rectangle, $\overrightarrow{AB}\perp\overrightarrow{BC}$ and $\overrightarrow{BC}\perp\overrightarrow{CD}$

$$= -\overrightarrow{AB}\bullet\overrightarrow{AB}+\overrightarrow{BC}\bullet\overrightarrow{BC} \qquad \text{Since } \overrightarrow{CD}=-\overrightarrow{AB}$$

$$= -\left\|\overrightarrow{AB}\right\|^2+\left\|\overrightarrow{BC}\right\|^2$$

Thus $\overrightarrow{AC}\bullet\overrightarrow{BD}=-\left\|\overrightarrow{AB}\right\|^2+\left\|\overrightarrow{BC}\right\|^2=0$ if and only if $\left\|\overrightarrow{AB}\right\|=\left\|\overrightarrow{BC}\right\|$, hence if and only if ABCD is a square.

13.



$$\begin{aligned}\left\|\overrightarrow{AC}\right\|^2+\left\|\overrightarrow{BD}\right\|^2 &= \left\|\overrightarrow{AB}+\overrightarrow{BC}\right\|^2+\left\|\overrightarrow{BC}+\overrightarrow{CD}\right\|^2\\ &=\left(\overrightarrow{AB}+\overrightarrow{BC}\right)\bullet\left(\overrightarrow{AB}+\overrightarrow{BC}\right)+\left(\overrightarrow{BC}+\overrightarrow{CD}\right)\bullet\left(\overrightarrow{BC}\bullet\overrightarrow{CD}\right)\\ &=\overrightarrow{AB}\bullet\overrightarrow{AB}+2\overrightarrow{AB}\bullet\overrightarrow{BC}+\overrightarrow{BC}\bullet\overrightarrow{BC}+\overrightarrow{BC}\bullet\overrightarrow{BC}+2\overrightarrow{BC}\bullet\overrightarrow{CD}+\overrightarrow{CD}\bullet\overrightarrow{CD}\\ &=\left\|\overrightarrow{AB}\right\|^2+2\left\|\overrightarrow{BC}\right\|^2+\left\|\overrightarrow{CD}\right\|^2+2\overrightarrow{AB}\bullet\overrightarrow{BC}+2\overrightarrow{BC}\bullet\overrightarrow{CD}\\ &=\left\|\overrightarrow{AB}\right\|^2+\left\|\overrightarrow{BC}\right\|^2+\left\|\overrightarrow{BC}\right\|^2+\left\|\overrightarrow{CD}\right\|^2+2\overrightarrow{BC}\bullet\left(\overrightarrow{AB}+\overrightarrow{CD}\right)\\ &=\left\|\overrightarrow{AB}\right\|^2+\left\|\overrightarrow{BC}\right\|^2+\left\|\overrightarrow{AD}\right\|^2+\left\|\overrightarrow{CD}\right\|^2+2\overrightarrow{BC}\bullet\left(\overrightarrow{AB}+\overrightarrow{BA}\right)\text{ since }\overrightarrow{BC}=\overrightarrow{AD}\text{ and }\overrightarrow{CD}=\overrightarrow{BA}\\ &=\left\|\overrightarrow{AB}\right\|^2+\left\|\overrightarrow{BC}\right\|^2+\left\|\overrightarrow{AD}\right\|^2+\left\|\overrightarrow{CD}\right\|^2\end{aligned}$$

14. a) $\text{proj}_{\vec{v}}\vec{u}=\left(\frac{-12}{13},\frac{18}{13}\right)$ $\text{proj}_{\vec{u}}\vec{v}=\left(\frac{-18}{25},\frac{-24}{25}\right)$

b) $\text{proj}_{\vec{v}}\vec{u}=\left(\frac{-6}{7},\frac{-15}{14},\frac{3}{14}\right)$ $\text{proj}_{\vec{u}}\vec{v}=\left(\frac{-9}{14},\frac{9}{7},\frac{-27}{14}\right)$

c) $\text{proj}_{\vec{v}}\vec{u}=\left(0,\frac{2}{9},\frac{5}{9},\frac{4}{9}\right)$ $\text{proj}_{\vec{u}}\vec{v}=\left(\frac{2}{5},0,\frac{-3}{10},\frac{1}{2}\right)$

d) $\text{proj}_{\vec{v}}\vec{u}=\left(\frac{-24}{25},\frac{6}{5},\frac{12}{25},\frac{12}{25}\right)$ $\text{proj}_{\vec{u}}\vec{v}=\left(0,\frac{4}{9},\frac{-4}{3},\frac{16}{9},\frac{4}{9}\right)$

15. $U\bullet AV=\left(AV\right)^T U=\left(V^T A^T\right)U=V^T\left(A^T U\right)=A^T U\bullet V$

16. $\vec{u}\bullet\left(\vec{v}+\vec{w}\right)=U\bullet\left(V+W\right)=\left(V+W\right)^T U=\left(V^T+W^T\right)U=V^T U+W^T U$

$=U\bullet V+U\bullet W=\vec{u}\bullet\vec{v}+\vec{u}\bullet\vec{w}$

17. a) $\cos\alpha=\frac{\hat{\imath}\bullet\vec{v}}{\|\hat{\imath}\|\|\vec{v}\|}=\frac{(1,0,0)\bullet(a,b,c)}{\|(1,0,0)\|\|\vec{v}\|}=\frac{a}{1\|\vec{v}\|}=\frac{a}{\|\vec{v}\|}$, $\cos\beta=\frac{b}{\|\vec{v}\|}$, $\cos\gamma=\frac{c}{\|\vec{v}\|}$

b) $\sqrt{\cos^2\alpha+\cos^2\beta+\cos^2\gamma}=\sqrt{\frac{a^2}{\|\vec{v}\|^2}+\frac{b^2}{\|\vec{v}\|^2}+\frac{c^2}{\|\vec{v}\|^2}}=\frac{1}{\|\vec{v}\|}\sqrt{a^2+b^2+c^2}=\frac{\|\vec{v}\|}{\|\vec{v}\|}=1$

c) $\|\vec{v}\|\left(\cos\alpha,\cos\beta,\cos\gamma\right)=\|\vec{v}\|\left(\frac{a}{\|\vec{v}\|},\frac{b}{\|\vec{v}\|},\frac{c}{\|\vec{v}\|}\right)=\left(a,b,c\right)=\vec{v}$

d) $\vec{v}_1\bullet\vec{v}_2=\|\vec{v}_1\|\left(\cos\alpha_1,\cos\beta_1,\cos\gamma_1\right)\bullet\|\vec{v}_2\|\left(\cos\alpha_2,\cos\beta_2,\cos\gamma_2\right)$

$=\|\vec{v}_1\|\|\vec{v}_2\|\left(\cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2\right)$

Thus if $\cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2=0$ then $\vec{v}_1\bullet\vec{v}_2=0$ so $\vec{v}_1$ and $\vec{v}_2$ are perpendicular.

e) $\vec{u}=4\left(\frac{\sqrt{3}}{2},0,\frac{-1}{2}\right)$, with direction cosines $\frac{\sqrt{3}}{2}$, 0 and $\frac{-1}{2}$ and direction angles $\frac{\pi}{6}$, $\frac{\pi}{2}$ and $\frac{2\pi}{3}$