MATHEMATICS 201-NYC-05
Vectors and Matrices
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IV - Determinants

1. Let $A=\begin{bmatrix}3 & -1 & 2\\ 5 & 7 & -2\\ 1 & 4 & 6\end{bmatrix}$. Find

   a) M_{21} b) m_{21} c) c_{21}
   d) M_{33} e) m_{33} f) c_{33}

2. Find the determinant of the following matrices. Make sure you try both methods, cofactor expansion and elementary row operations.

   a) $\begin{bmatrix}5 & 9\\ -1 & 3\end{bmatrix}$ b) $\begin{bmatrix}5 & 2\\ 3 & -7\end{bmatrix}$ c) $\begin{bmatrix}1 & -5 & 2\\ 6 & -3 & 1\\ 4 & 6 & 3\end{bmatrix}$

   d) $\begin{bmatrix}-2 & 0 & 3\\ 1 & 10 & -3\\ 1 & 4 & 5\end{bmatrix}$ e) $\begin{bmatrix}1 & -4 & 0\\ 3 & 5 & 2\\ 0 & 3 & 9\end{bmatrix}$ f) $\begin{bmatrix}2 & -1 & 3\\ 0 & 0 & 5\\ 2 & -2 & 4\end{bmatrix}$

   g) $\begin{bmatrix}3 & 6 & -5 & 4\\ -2 & 0 & 6 & 0\\ 1 & 1 & 2 & 2\\ 0 & 3 & -1 & -1\end{bmatrix}$ h) $\begin{bmatrix}3 & 9 & -2 & 1\\ 4 & 5 & -8 & 3\\ 0 & 0 & 0 & 0\\ 8 & -6 & 1 & 3\end{bmatrix}$ i) $\begin{bmatrix}5 & 2 & 0 & 0 & -2\\ 0 & 1 & 4 & 3 & 2\\ 0 & 0 & 2 & 6 & 3\\ 0 & 0 & 3 & 4 & 1\\ 0 & 0 & 0 & 0 & 2\end{bmatrix}$

3. Consider the matrices $A=\begin{bmatrix}1 & -7 & 5\\ 3 & -2 & 2\\ 4 & 0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 3 & -2\\ 4 & 0 & -3\\ -7 & 5 & 4\end{bmatrix}$.

   Evaluate, using the properties of the determinant if possible.
   a) $\det(A)$
   b) $\det(B)$.
   c) $\det(AB)$.
   d) $\det(A^{-1})$.
   e) $\det(A^{T})$.
   f) $\det(A+B)$.

4. Let A and B be 3×3 matrices such that $\det(A)=4$ and $\det(B)=-3$. Evaluate the following.

a) $\det(AB)$ b) $\det\left(A^{-1}\right)$ c) $\det(A^T)$

d) $\det(4B)$ e) $\det\left((AB)^T\right)$ f) $\det\left((AB)^{-1}\right)$

g) $\det\left(B^6\right)$ h) $\det\left(A^{-1}B^T\right)$

5. If A is an $n\times n$ matrix such that $\det(-A)=\det(A)$, what can you say about n?

6. Let A be a matrix such that $A^2=A$. Prove that either A is singular or $\det(A)=1$.

7. Prove that if A in an invertible matrix, then so is A^TA.

8. Prove that if P is an invertible matrix, then $\det(A)=\det\left(P^{-1}AP\right)$.

9. Show that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$$

This is called a **Vandermonde** determinant.

10. Show that if $a\neq 0$ $b\neq 0$ $c\neq 0$ then

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

11. Find all values of t such that $A=\begin{bmatrix} 2 & 2 & 1 \\ 3 & 1 & t \\ 1 & t & -2 \end{bmatrix}$ is invertible.

12. For each of the given matrix A, find the eigenvalues and a corresponding eigenvector.

a) $A=\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ b) $A=\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

c) $A=\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ d) $A=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 6 & 0 & 2 \end{bmatrix}$ e) $A=\begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

13. Let A be a 2×2 matrix. Show that the characteristic equation of A is given by

$$\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0$$

14. Let f and g be two differentiable functions, and the ***Wronskian matrix*** $W(x)$ defined by

$$W(x)=\begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix}$$

Find $\det(W(x))$ if

a) $f(x)=1$ and $g(x)=x$

b) $f(x)=e^{ax}$ and $g(x)=\ln(bx)$

b) $f(x)=\sin 2x$ and $g(x)=\cos 3x$

c) $f(x)=\dfrac{1}{x}$ and $g(x)=\dfrac{1}{x^2}$

ANSWERS

1. a) $M_{21}=\begin{bmatrix} -1 & 2 \\ 4 & 6 \end{bmatrix}$ b) $m_{21}=-14$ c) $a_{21}=14$

   d) $M_{33}=\begin{bmatrix} 3 & -1 \\ 5 & 7 \end{bmatrix}$ e) $m_{33}=26$ f) $a_{33}=26$

2. a) 24 b) -41 c) 151 d) –142 e) 147 f)10

   g) –108 h) 0 i) -100

3. a) $\det(A)=3$ b) $\det(B)=-10$ c) $\det(AB)=-30$

   d) $\det(A^{-1})=\dfrac{1}{3}$ e) $\det(A^T)=3$ f) $\det(A+B)=205$

4. a) $\det(AB)=-12$ b) $\det\left(A^{-1}\right)=\frac{1}{4}$ c) $\det(A^T)=4$

   d) $\det(4B)=-192$ e) $\det\left((AB)^T\right)=-12$ f) $\det\left((AB)^{-1}\right)=-\frac{1}{12}$

   g) $\det\left(B^6\right)=729$ h) $\det\left(A^{-1}B^T\right)=-\frac{3}{4}$

5. n is an even number

6. A is either singular or invertible.

   (i) If A is singular then $\det(A)=0$.

   (ii) If A is invertible, then

$$A^2=A$$
$$A^{-1}AA=A^{-1}A$$
$$IA=I$$
$$A=I$$

   Thus $\det(A)=\det(I)=1$

   Hence either A is singular or $\det(A)=1$

7. If A is invertible, then $\det(A)\neq 0$ so

$$\det\left(AA^T\right)=\det(A)\det\left(A^T\right)=\det(A)\det(A)=\left[\det(A)\right]^2\neq 0$$

   hence A^TA is invertible.

8. If P is invertible, then $\det\left(P\right)\neq 0$ so

$$\det\left(P^{-1}AP\right)=\det\left(P^{-1}\right)\det\left(A\right)\det\left(P\right)=\frac{1}{\det\left(P\right)}\det\left(A\right)\det\left(P\right)=\det\left(A\right)$$

hence $\det\left(A\right)=\det\left(P^{-1}AP\right)$.

9. $$\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}\overset{R_2\to R_2-R_1}{=}\begin{vmatrix}1 & a & a^2\\ 0 & b-a & b^2-a^2\\ 1 & c & c^2\end{vmatrix}\overset{R_3\to R_3-R_1}{=}\begin{vmatrix}1 & a & a^2\\ 0 & b-a & b^2-a^2\\ 0 & c-a & c^2-a^2\end{vmatrix}$$

$$\begin{aligned}&=1\cdot\left[\left(b-a\right)\left(c^2-a^2\right)-\left(c-a\right)\left(b^2-a^2\right)\right]\\&=(b-a)(c-a)(c+a)-(c-a)(b-a)(b+a)\\&=(b-a)(c-a)\left[(c+a)-(b+a)\right]\\&=(b-a)(c-a)(c-b)\end{aligned}$$

10. $$\begin{vmatrix}1+a & 1 & 1\\ 1 & 1+b & 1\\ 1 & 1 & 1+c\end{vmatrix}=\left(1+a\right)\left[\left(1+b\right)\left(1+c\right)-1\right]-\left(1+c-1\right)+\left[1-\left(1+b\right)\right]$$

$$\begin{aligned}&=\left(1+a\right)\left(b+c+bc\right)-c-b\\&=b+c+bc+ab+ac+abc-c-b\\&=abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\end{aligned}$$

11. $t\neq -1,\frac{7}{2}$

12. a) $\lambda_1=2\quad v_1=\left(1,0\right)$, $\quad\lambda_2=3\quad v_2=\left(1,1\right)$
 b) $\lambda_1=0\quad v_1=\left(1,2\right)$, $\quad\lambda_2=7\quad v_2=\left(-3,1\right)$
 c) $\lambda_1=1\quad v_1=\left(-1,-2,1\right)$, $\quad\lambda_2=2\quad v_2=\left(1,0,0\right)$, $\quad\lambda_3=3\quad v_3=\left(0,1,0\right)$
 d) $\lambda_1=-1\quad v_1=\left(1,0,-2\right)$, $\quad\lambda_2=0\quad v_2=\left(0,1,0\right)$, $\quad\lambda_3=4\quad v_3=\left(1,0,3\right)$
 e) $\lambda_1=-1\quad v_1=\left(2,-1,1\right)$, $\quad\lambda_2=1\quad v_2=\left(-2,1,1\right)$, $\quad\lambda_3=3\quad v_3=\left(-2,-1,1\right)$

13. Let $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$. Thus $\operatorname{tr}\left(A\right)=a+d$ and $\det\left(A\right)=ad-bc$

Hence $\det\left(\lambda I-A\right)=\begin{vmatrix}\lambda-a & -b\\ -c & \lambda-d\end{vmatrix}=\left(\lambda-a\right)\left(\lambda-d\right)-bc$

$$\begin{aligned}&=\lambda^2-\left(a+d\right)\lambda+ad-bc\\&=\lambda^2-\operatorname{tr}(A)\lambda+\det(A)\end{aligned}$$

14. a) 1 b) $\dfrac{e^{ax}\left(1-ax\ln\left(bx\right)\right)}{x}$ c) $-3\sin\left(2x\right)\sin\left(3x\right)-2\cos\left(2x\right)\cos\left(3x\right)$ d) $-\frac{1}{x^4}$