MATHEMATICS 201-NYC-05
Vectors and Matrices
Martin Huard
Fall 2007

III - Inverse and Elementary Matrices

1. Show that B is the inverse of A.

   a) $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, $B=\begin{bmatrix}-2 & 1\\ \frac{3}{2} & -\frac{1}{2}\end{bmatrix}$

   b) $A=\begin{bmatrix}-2 & 2 & 3\\ 1 & -1 & 0\\ 0 & 1 & 4\end{bmatrix}$, $B=\frac{1}{3}\begin{bmatrix}-4 & -5 & 3\\ -4 & -8 & 3\\ 1 & 2 & 0\end{bmatrix}$

2. Find the inverse of the matrix (if it exists).

   a) $\begin{bmatrix}1 & 2\\ 3 & 7\end{bmatrix}$ b) $\begin{bmatrix}-7 & 33\\ 4 & -19\end{bmatrix}$ c) $\begin{bmatrix}2 & 4\\ 4 & 8\end{bmatrix}$

   d) $\begin{bmatrix}1 & 1 & 1\\ 3 & 5 & 4\\ 3 & 6 & 5\end{bmatrix}$ e) $\begin{bmatrix}1 & 1 & 2\\ 3 & 1 & 0\\ -2 & 0 & 3\end{bmatrix}$ f) $\begin{bmatrix}1 & 0 & 0\\ 3 & 4 & 0\\ 2 & 5 & 5\end{bmatrix}$

3. Use an inverse matrix to solve the given systems of linear equations.

   a) $\begin{aligned}-x+y&=4\\ -2x+y&=0\end{aligned}$ b) $\begin{aligned}-x+y&=-3\\ -2x+y&=5\end{aligned}$ c) $\begin{aligned}-x+y&=0\\ -2x+y&=0\end{aligned}$

4. Use an inverse matrix to solve the given systems of linear equations.

   a) $\begin{aligned}3x+2y+2z&=0\\ 2x+2y+2z&=5\\ -4x+4y+3z&=2\end{aligned}$ b) $\begin{aligned}3x+2y+2z&=-1\\ 2x+2y+2z&=2\\ -4x+4y+3z&=0\end{aligned}$ c) $\begin{aligned}3x+2y+2z&=0\\ 2x+2y+2z&=0\\ -4x+4y+3z&=0\end{aligned}$

5. Using the following matrices,

   $A^{-1}=\begin{bmatrix}2 & 5\\ -7 & 6\end{bmatrix}$, $B^{-1}=\begin{bmatrix}7 & -3\\ 2 & 0\end{bmatrix}$

   find the following, using the properties of the inverse.

   a) $(AB)^{-1}$ b) $(A^T)^{-1}$ c) A^{-2} d) $(2A)^{-1}$

6. Find A such that

   a) $(2A)^{-1}=\begin{bmatrix}2 & -1\\ 3 & 5\end{bmatrix}$ b) $(I+3A)^{-1}=\begin{bmatrix}3 & 4\\ -1 & 2\end{bmatrix}$

7. Prove that if C is an invertible matrix such that $CA = CB$, then $A = B$.

8. Determine whether the matrix is elementary. If it is, state the elementary row operation that was used to produce it.

a) $\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ e) $\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

9. Let A, B and C be given by

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 3 & 3 \\ -2 & -4 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -2 & -4 & 5 \\ 1 & 3 & 3 \\ 2 & 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & -4 & 4 \\ 1 & 3 & 3 \\ -2 & -4 & 5 \end{bmatrix}$$

a) Find an elementary matrix E such that $EA = B$.
b) Find an elementary matrix E such that $EA = C$.
c) Find an elementary matrix E such that $EB = A$.
d) Find an elementary matrix E such that $EC = A$.

10. Find the inverse of the given elementary matrix.

a) $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $k \neq 0$

11. For each of the given matrices A, factor A^{-1} and A into a product of elementary matrices.

a) $\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ c) $\begin{bmatrix} -1 & 2 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ d) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}$

12. Prove that if B is invertible and $AB = BA$, then $AB^{-1} = B^{-1}A$.

13. Show that $(I - A)^{-1} = I + A + A^2 + A^3$ if $A^4 = 0$.

14. Show that there are no matrices A and B such that $AB - BA = I$. (*Hint*: take the trace of both sides of the equation).

15. Assuming the stated inverses exist, prove the following identities.

a) $C(C + D)^{-1} D = (C^{-1} + D^{-1})^{-1}$

b) $(I + CD)^{-1} C = C(I + DC)^{-1}$

Answers

1. a) $AB=\begin{bmatrix}1&2\\3&4\end{bmatrix}\begin{bmatrix}-2&1\\\frac{3}{2}&-\frac{1}{2}\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}=I$ $\quad BA=\begin{bmatrix}-2&1\\\frac{3}{2}&-\frac{1}{2}\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}=I$

   b) $AB=\frac{1}{3}\begin{bmatrix}-2&2&3\\1&-1&0\\0&1&4\end{bmatrix}\begin{bmatrix}-4&-5&3\\-4&-8&3\\1&2&0\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I$

   $BA=\frac{1}{3}\begin{bmatrix}-4&-5&3\\-4&-8&3\\1&2&0\end{bmatrix}\begin{bmatrix}-2&2&3\\1&-1&0\\0&1&4\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I$

2. a) $\begin{bmatrix}7&-2\\-3&1\end{bmatrix}$ b) $\begin{bmatrix}-19&-33\\-4&-7\end{bmatrix}$ c) Does not exist

   d) $\begin{bmatrix}1&1&-1\\-3&2&-1\\3&-3&2\end{bmatrix}$ e) $\begin{bmatrix}-\frac{3}{2}&\frac{3}{2}&1\\\frac{9}{2}&-\frac{7}{2}&-3\\-1&1&1\end{bmatrix}$ f) $\begin{bmatrix}1&0&0\\-\frac{3}{4}&\frac{1}{4}&0\\\frac{7}{20}&-\frac{1}{4}&\frac{1}{5}\end{bmatrix}$

3. a) $x=4,\ y=8$ b) $x=-8,\ y=-11$ c) $x=0,\ y=0$
4. a) $x=-5,\ y=-\frac{81}{2},\ z=48$ b) $x=-3,\ y=-24,\ z=28$ c) $x=0,\ y=0,\ z=0$
5. a) $\begin{bmatrix}35&17\\4&10\end{bmatrix}$ b) $\begin{bmatrix}2&-7\\5&6\end{bmatrix}$ c) $\begin{bmatrix}-31&40\\-56&1\end{bmatrix}$ d) $\begin{bmatrix}1&\frac{5}{2}\\-\frac{7}{2}&3\end{bmatrix}$
6. a) $\begin{bmatrix}\frac{5}{26}&\frac{1}{26}\\-\frac{3}{26}&\frac{1}{13}\end{bmatrix}$ b) $\begin{bmatrix}-\frac{4}{15}&-\frac{2}{15}\\\frac{1}{30}&-\frac{7}{30}\end{bmatrix}$
7. We have $CA=CB$. Since C is invertible, then multiplying by C^{-1}, we have $C^{-1}CA=C^{-1}CB$

   $IA=IB$

   $A=B$
8. a) Yes: $R_2\to\sqrt{2}R_2$ b) No c) No d) Yes: $R_2\to R_2+2R_3$ e) No f) No
9. a) $E=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$ b) $E=\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$ c) $E=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$ d) $E=\begin{bmatrix}1&0&-1\\0&1&0\\0&0&1\end{bmatrix}$
10. a) $\begin{bmatrix}1&0&2\\0&1&0\\0&0&1\end{bmatrix}$ b) $\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$ c) $\begin{bmatrix}1&0&0\\0&\frac{1}{k}&0\\0&0&1\end{bmatrix}$
11. a) $A^{-1}=\begin{bmatrix}1&0\\-4&1\end{bmatrix}\begin{bmatrix}\frac{1}{2}&0\\0&1\end{bmatrix}$ $\quad A=\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}1&0\\4&1\end{bmatrix}$

    b) $A^{-1}=\begin{bmatrix}1&-1\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1&0\\-2&1\end{bmatrix}$ $\quad A=\begin{bmatrix}1&0\\2&1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix}$

c)
$$A^{-1}=\begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&\frac{1}{2}&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$
$$A=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&2&1\end{bmatrix}\begin{bmatrix}1&-2&0\\0&1&0\\0&0&1\end{bmatrix}$$

d)
$$A^{-1}=\begin{bmatrix}1&-2&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-3\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$$
$$A=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&3\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix}$$

12. We have $AB=BA$. Since B is invertible, multiplying both sides, on the left and on the right, by B^{-1} we obtain
$$B^{-1}ABB^{-1}=B^{-1}BAB^{-1}$$
$$B^{-1}AI=IAB^{-1}$$
$$B^{-1}A=AB^{-1}$$
which is equivalent to $AB^{-1}=B^{-1}A$.

13. Let us show that the inverse of $I-A$ is $I+A+A^2+A^3$ if $A^4=0$.
$$\begin{aligned}(I-A)\left(I+A+A^2+A^3\right)&=II+IA+IA^2+IA^3-AI-AA-AA^2-AA^3\\&=I+A+A^2+A^3-A-A^2-A^3-A^4\\&=I \qquad (\text{ Since } A^4=0\text{ })\end{aligned}$$

14. Suppose $AB-BA=I$, then $tr(AB-BA)=tr(I)$. But since $tr(AB-BA)=tr(AB)-tr(BA)=tr(AB)-tr(AB)=0$ and $tr(I)=n$, we have a contradiction, thus we cannot have $AB-BA=I$.

15. a)
$$\begin{aligned}C(C+D)^{-1}D&=\left[(C+D)C^{-1}\right]^{-1}D\\&=\left(CC^{-1}+DC^{-1}\right)^{-1}D\\&=\left(I+DC^{-1}\right)^{-1}D\\&=\left[D^{-1}\left(I+DC^{-1}\right)\right]^{-1}\\&=\left(D^{-1}I+D^{-1}DC^{-1}\right)^{-1}\\&=\left(D^{-1}+IC^{-1}\right)^{-1}\\&=\left(D^{-1}+C^{-1}\right)^{-1}\end{aligned}$$

b)
$$\begin{aligned}(I+CD)^{-1}C&=\left[C^{-1}(I+CD)\right]^{-1}\\&=\left(C^{-1}I+C^{-1}CD\right)^{-1}\\&=\left(C^{-1}+ID\right)^{-1}\\&=\left(C^{-1}+DI\right)^{-1}\\&=\left(C^{-1}+DCC^{-1}\right)^{-1}\\&=\left[(I+DC)C^{-1}\right]^{-1}\\&=C(I+DC)^{-1}\end{aligned}$$