MATHEMATICS 201-NYC-05
Vectors and Matrices
Martin Huard
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II - Systems of Linear Equations

1. Which of the following are linear equations?
   a) $x_1 + 3x_2 - 5x_3 = \sqrt{5}$ b) $x_1 + 3x_1x_2 - 5x_3 = 1$ c) $x_1 = 3x_2 + \pi$

2. Find a solution set for each of the following linear equations.
   a) $2x - 4y = 5$ b) $4x_1 - 2x_2 + 3x_3 = 2$ c) $3v + 2w - 2x - y + z = 0$

3. Write the following systems of linear equations in the form $AX = b$ and give the augmented matrix.

   a) $\begin{aligned} 2x - 3y &= 7 \\ 4x + 5y &= 3 \end{aligned}$

   b) $\begin{aligned} 2x + 3y - z &= 4 \\ -3x - y + z &= 1 \\ x + 5y &= 7 \end{aligned}$

   c) $\begin{aligned} a + 6b - c + d &= 2 \\ 3b + c - 3 + 2a &= 5 \\ c - d + a - 3b &= 1 \\ d - 3a + 4b + c &= 3 \end{aligned}$

   d) $\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$

4. Find a system of linear equations corresponding to the following augmented matrices.

   a) $\begin{bmatrix} 3 & 2 & -4 \\ 2 & -1 & 4 \\ -3 & 0 & 6 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 3 & 0 & -1 & 0 \\ 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 5 & 0 & 4 \end{bmatrix}$

5. Determine whether the matrix is in row-echelon form, reduced row-echelon form or neither.

   a) $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

   d) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ e) $\begin{bmatrix} 1 & 5 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ f) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

6. Solve the following systems of linear equations using Gaussian elimination.

   a) $\begin{aligned} x + 2y &= 7 \\ 2x + y &= 8 \end{aligned}$

   b) $\begin{aligned} x - 3y &= 5 \\ -2x + 6y &= -10 \end{aligned}$

   c) $\begin{aligned} -x + 2y &= 1.5 \\ 2x - 4y &= 3 \end{aligned}$

d) $\begin{aligned} -3x+5y &= -2 \\ 3x+4y &= 4 \\ 4x-8y &= 32 \end{aligned}$

e) $\begin{aligned} x_1 - 3x_3 &= -2 \\ 3x_1 + x_2 - 2x_3 &= 5 \\ 2x_1 + 2x_2 + x_3 &= 4 \end{aligned}$

f) $\begin{aligned} x_1 + x_2 - 5x_3 &= 3 \\ x_1 - 2x_3 &= 1 \\ 2x_1 - x_2 - x_3 &= 0 \end{aligned}$

g) $\begin{aligned} x + 2y + z &= 8 \\ -3x - 6y - 3z &= -21 \end{aligned}$

h) $\begin{aligned} 2x + y + 2z &= 7 \\ x + y + 2z &= 4 \\ 2x - 2y - 4z &= 4 \end{aligned}$

i) $\begin{aligned} x - 2y + 4z &= 6 \\ 2x - 4y + 10z &= 18 \\ -3x + 6y - 9z &= -9 \end{aligned}$

j) $\begin{aligned} 3x + 3y + 12z &= 6 \\ x + y + 4z &= 2 \\ 2x + 5y + 20z &= 10 \\ -x + 2y + 8z &= 4 \end{aligned}$

k) $\begin{aligned} 2x - 3y + 4z - t &= 6 \\ x + y - 3z + 2t &= -2 \\ x - 4y + 7z - 3t &= 8 \\ 4x - y - 2z + 3t &= 2 \end{aligned}$

l) $\begin{aligned} x - 2t &= 3 \\ 3x - y - 3z + 10t &= 4 \\ 4x + y + 3z - 11t &= 17 \\ 5x + 3y + 9z + 12t &= 30 \end{aligned}$

7. Solve the systems of linear equations in 6 using the Gauss-Jordan method.

8. Solve the following homogeneous systems of linear equations by any method.

a) $\begin{aligned} x - y + z &= 0 \\ x + y &= 0 \\ x + 2y - z &= 0 \end{aligned}$

b) $\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ -2x_1 - 2x_2 - 2x_3 &= 0 \\ 3x_1 + 3x_2 + 3x_3 &= 0 \end{aligned}$

c) $\begin{aligned} x + 2y + z + w &= 0 \\ x - y + w &= 0 \\ y - z + 2w &= 0 \end{aligned}$

9. Find A such that

$$A \bullet \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

10. Solve the following (non-linear) systems of equations.

a) $\begin{aligned} 3x^2 + y^2 + 4z^2 &= -1 \\ 2x^2 + y^2 + 2z^2 &= 0 \\ 5x^2 + 2y^2 + 7z^2 &= -4 \end{aligned}$

b) $\begin{aligned} 2e^x + \ln t + 3\sin\theta &= 5 \\ 4e^x + \ln t + 5\sin\theta &= 9 \\ -2e^x + 3\ln t + 4\sin\theta &= 2 \end{aligned}$

11. Find values of a, b and c (if possible) such that the given system of linear equations has
    i) a unique solution
    ii) no solution
    iii) an infinite number of solutions

a) $\begin{aligned} x + y &= 2 \\ y + z &= 2 \\ x + z &= 2 \\ ax + by + cz &= 0 \end{aligned}$

b) $\begin{aligned} x + y &= 0 \\ y + z &= 0 \\ x + z &= 0 \\ ax + by + cz &= 0 \end{aligned}$

12. For which values of a will the following system of linear equations have

$$\begin{aligned} x + 2y - \quad & 3z = \quad 4 \\ 3x - \quad y + \quad & 5z = \quad 2 \\ 4x + \quad y + & \left(a^2 - 14\right) z = a + 2 \end{aligned}$$

i) a unique solution
ii) no solution
iii) an infinite number of solutions

13. For which values of k will the following system of linear equations have

$$\begin{aligned} x - \quad y + 2z &= \quad 0 \\ y - \quad z &= k^2 \\ -x + 2y - 3z &= \quad 1 \end{aligned}$$

i) a unique solution
ii) no solution
iii) an infinite number of solutions

14. For which values of a will the following system of linear equations have

$$\begin{aligned} a^2 x + 2y + \quad 2z &= 0 \\ x + \quad y + \quad 2z &= 1 \\ -x + \quad y + 2az &= 3 \end{aligned}$$

i) a unique solution
ii) no solution
iii) an infinite number of solutions

15. What condition on a, b and c would ensure that the following system of linear equations is consistent?

$$\begin{aligned} 2x + 3y - \quad 5z &= a \\ 5x - 2y + \quad 3z &= b \\ x - 8y + 13z &= c \end{aligned}$$

ANSWERS

1. a, c
2. a) $\left(\frac{5}{2}+2t,t\right)$ b) $\left(\frac{1}{2}+\frac{1}{2}s-\frac{3}{4}t,s,t\right)$ c) $\left(-\frac{2}{3}d+\frac{2}{3}c+\frac{1}{3}b-\frac{1}{3}a,d,c,b,a\right)$
3. a) $\begin{bmatrix}2&-3\\4&5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}7\\3\end{bmatrix}$ $\left[\begin{array}{cc|c}2&-3&7\\4&5&3\end{array}\right]$

   b) $\begin{bmatrix}2&3&-1\\-3&-1&1\\1&5&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4\\1\\7\end{bmatrix}$ $\left[\begin{array}{ccc|c}2&3&-1&4\\-3&-1&1&1\\1&5&0&7\end{array}\right]$

   c) $\begin{bmatrix}1&6&-1&1\\2&3&1&0\\1&-3&1&-1\\-3&4&1&1\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}=\begin{bmatrix}2\\8\\1\\3\end{bmatrix}$ $\left[\begin{array}{cccc|c}1&6&-1&1&2\\2&3&1&0&8\\1&-3&1&-1&1\\-3&4&1&1&3\end{array}\right]$

   d) $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ $\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right]$

4. a) $\begin{aligned}3x+2y&=-4\\2x-y&=4\\-3x&=6\end{aligned}$ b) $\begin{aligned}2x+3y-w&=0\\3z+2w&=1\\5z&=4\end{aligned}$
5. Row-echelon form : a, c, d, e, f Reduced row-echelon form : c, d, f
6. a) $(3,2)$ b) $(5+3t,t)$ c) No solution
   d) No solution e) $(4,-3,2)$ f) $(1+2t,2+3t,t)$
   g) No solution h) $(3,1-2t,t)$ i) $(2t-6,t,3)$
   j) $(0,2-4t,t)$ k) $(s-r,2s-r-2,s,r)$ l) $(3,5-3s,s,0)$
8. a) $(0,0,0)$ b) $(-r-s,r,s)$ c) $\left(\frac{-3}{2}r,\frac{-1}{2}r,\frac{3}{2}r,r\right)$
9. $A=\begin{bmatrix}\frac{1}{17}&\frac{3}{17}\\-\frac{5}{17}&\frac{2}{17}\end{bmatrix}$
10. a) No solution b) $x=0$, $t=1$ and $\theta=\frac{\pi}{2}+2\pi n$ $n\in\mathbb{Z}$
11. a) (i) $a+b+c=0$ ii) $a+b+c\neq 0$ iii) Never
    b) (i) always ii) never iii) never
12. i) $a\neq\pm 4$ ii) $a=-4$ iii) $a=4$
13. i) never ii) $k\neq\pm 1$ iii) $k=\pm 1$
14. i) $a\neq -1,0,2$ ii) $a=-1,0$ iii) $a=2$
15. $2a-b+c=0$