MATHEMATICS 201-NYC-05
Vectors and Matrices
Martin Huard
Fall 2007

I - Matrices

1. Let $A=\begin{bmatrix} 2 & -3 & 5 \\ 1 & \sqrt{2} & 7 \\ \pi & 3 & 0 \end{bmatrix}$. Find a_{23}, a_{12} and a_{31}.

2. Consider the following matrices.

$$A=\begin{bmatrix} 2 & -3 & 5 \\ 0 & 7 & 7 \\ 0 & 0 & 1 \end{bmatrix} \quad B=\begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ 0 & 3 & 2 \end{bmatrix} \quad C=\begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad D=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$E=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad F=\begin{bmatrix} 3 & -2 & 4 \\ -2 & 0 & 3 \\ 4 & 3 & 5 \end{bmatrix} \quad G=\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \quad H=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Determine which of these matrices (if any) are

a) lower triangular b) upper triangular c) square
d) diagonal e) column f) row
g) symmetric

3. Consider the following matrices.

$$A=\begin{bmatrix} -1 & -3 \\ 4 & 2 \\ 2 & -1 \end{bmatrix} \quad B=\begin{bmatrix} 2 & -3 \\ 4 & -2 \end{bmatrix} \quad C=\begin{bmatrix} 3 & 4 & 2 \\ -2 & 4 & -3 \end{bmatrix} \quad D=\begin{bmatrix} 5 & -1 & 2 \\ 2 & 0 & -4 \\ 1 & 3 & -2 \end{bmatrix} \quad E=\begin{bmatrix} 2 & -1 & 0 \\ 5 & -2 & 5 \\ 2 & 1 & 1 \end{bmatrix}$$

Compute the following (where possible).

a) $D+E$ b) $D-E$ c) $3C$ d) $-2A$
e) $2D+4E$ f) $3B-C$ g) $-2(2E-4D)$ h) $E-E$
i) $\operatorname{tr}(D)$ j) $\operatorname{tr}(C)$ k) $\operatorname{tr}(2D-5E)$ l) $\operatorname{tr}(5E)$

4. Using the matrices defined in 3, compute the following.

a) $2B^T$ b) C^T-A c) $2E^T-3D$ d) $\left(2A^T-3C\right)^T$
e) AB f) BA g) AC h CA
i) $C(DE)$ j) C^TB k) BD l) $\operatorname{tr}(B-A^TC^T)$
m) $(DE)^T-D^TE^T$ n) AA^T o) D^2 p) D^3

5. Consider the matrix $A=\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$. Find A^2, A^3, A^n.

6. Find x and y such that

$$\begin{bmatrix} x & 3 \\ -6 & y \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

7. Solve for A.

a) $\begin{bmatrix} 2 & -2 \\ 3 & 1 \end{bmatrix} - 4A = \begin{bmatrix} 3 & 5 \\ -2 & 7 \end{bmatrix}$

b) $2A - 3\begin{bmatrix} 3 & -2 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$

8. Let A and B be two square matrices. Prove that

a) $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$

b) $\mathrm{tr}(kA) = k \cdot \mathrm{tr}(A)$

c) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ (If A and B are 3×3 matrices.)

9. Prove that if A is a square matrix, then $A + A^T$ is symmetric.

10. Let A be a symmetric matrix.

a) Show that A^2 is symmetric.

b) Show that $3A^2 + 2A - 3I$ is symmetric.

11. Prove that if $A^T A = A$, then

a) A is symmetric

b) $A = A^2$

12. Let f and g be two differentiable functions, and the ***Wronskian matrix*** $W(x)$ defined by

$$W(x) = \begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix}$$

Find $W(x)$ if

a) $f(x) = 1$ and $g(x) = x$

b) $f(x) = e^{ax}$ and $g(x) = \ln(bx)$

b) $f(x) = \sin 2x$ and $g(x) = \cos 3x$

c) $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{1}{x^2}$

13. Let A and B be two matrices defined by

$$A = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \quad \text{and } B = I - A$$

a) Prove that $A^2 = A$ and $B^2 = B$

b) Evaluate AB and BA.

14. A matrix B is said to be the ***square root*** of a matrix A if $B^2 = A/$

a) Find two square roots of $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

b) Find all square roots of $\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$.

Answers

1. 7, -3, π
2. a) B, C, H b) A, H c) A, B, C, F, H d) H
   e) G f) None g) F, H
3. a) $\begin{bmatrix} 7 & -2 & 2 \\ 7 & -2 & 1 \\ 3 & 4 & -1 \end{bmatrix}$ b) $\begin{bmatrix} 3 & 0 & 2 \\ -3 & 2 & -9 \\ -1 & 2 & -3 \end{bmatrix}$ c) $\begin{bmatrix} 9 & 12 & 6 \\ -6 & 12 & -9 \end{bmatrix}$

   d) $\begin{bmatrix} 2 & 6 \\ -8 & -4 \\ -4 & 2 \end{bmatrix}$ e) $\begin{bmatrix} 18 & -6 & 4 \\ 24 & -8 & 12 \\ 10 & 10 & 0 \end{bmatrix}$ f) Undefined g) $\begin{bmatrix} 32 & -4 & 16 \\ -4 & 8 & -52 \\ 0 & 20 & -20 \end{bmatrix}$

   h) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ i) 3 j) Undefined k) 1 l) 5
4. a) $\begin{bmatrix} 4 & 8 \\ -6 & -4 \end{bmatrix}$ b) $\begin{bmatrix} 4 & 1 \\ 0 & 2 \\ 0 & -2 \end{bmatrix}$ c) $\begin{bmatrix} -11 & 13 & -2 \\ -8 & -4 & 14 \\ -3 & 1 & 8 \end{bmatrix}$ d) $\begin{bmatrix} -11 & 0 \\ -4 & -8 \\ -2 & 7 \end{bmatrix}$

   e) $\begin{bmatrix} -14 & 9 \\ 16 & -16 \\ 0 & -4 \end{bmatrix}$ f) Undefined g) $\begin{bmatrix} 3 & -16 & 7 \\ 8 & 24 & 2 \\ 8 & 4 & 7 \end{bmatrix}$ h) $\begin{bmatrix} 17 & -3 \\ 12 & 17 \end{bmatrix}$

   i) $\begin{bmatrix} 37 & -45 & 1 \\ -73 & 5 & -49 \end{bmatrix}$ j) $\begin{bmatrix} -2 & -5 \\ 24 & -20 \\ -8 & 0 \end{bmatrix}$ k) Undefined l) –34

   m) $\begin{bmatrix} 1 & -30 & 0 \\ 1 & -16 & -10 \\ -11 & -12 & 15 \end{bmatrix}$ n) $\begin{bmatrix} 10 & -10 & 1 \\ -10 & 20 & 6 \\ 1 & 6 & 5 \end{bmatrix}$ o) $\begin{bmatrix} 25 & 1 & 10 \\ 6 & -14 & 12 \\ 9 & -7 & -6 \end{bmatrix}$

   p) $\begin{bmatrix} 137 & 5 & 26 \\ 14 & 30 & 44 \\ 25 & -27 & 58 \end{bmatrix}$
5. $A^2 = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}$ $A^3 = \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}$ $A^n = \begin{bmatrix} a^n & na^{n-1}b \\ 0 & a^n \end{bmatrix}$
6. $x = -\frac{7}{2}$ $y = \frac{9}{4}$
7. a) $A = \begin{bmatrix} \frac{-1}{4} & \frac{-7}{4} \\ \frac{5}{4} & \frac{-3}{2} \end{bmatrix}$ b) $A = \begin{bmatrix} \frac{5}{2} & \frac{-1}{2} \\ 11 & 8 \end{bmatrix}$

8. a) $\operatorname{tr}(A+B)=\left(a_{11}+b_{11}\right)+\left(a_{22}+b_{22}\right)+\ldots+\left(a_{nn}+b_{nn}\right)$

$$=a_{11}+a_{22}+\ldots+a_{nn}+b_{11}+b_{22}+\ldots+b_{nn}$$

$$=\operatorname{tr}(A)+\operatorname{tr}(B)$$

b) $\operatorname{tr}(kA)=ka_{11}+ka_{22}+\ldots+ka_{nn}=k\left(a_{11}+a_{22}+\ldots+a_{nn}\right)=k\cdot\operatorname{tr}(A)$

c) $\operatorname{tr}(AB)=\left(a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}\right)+\left(a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}\right)+\left(a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33}\right)$

$$=\left(b_{11}a_{11}+b_{21}a_{12}+b_{31}a_{13}\right)+\left(b_{12}a_{21}+b_{22}a_{22}+b_{32}a_{23}\right)+\left(b_{13}a_{31}+b_{23}a_{32}+b_{33}a_{33}\right)$$

$$=\left(b_{11}a_{11}+b_{12}a_{21}+b_{13}a_{31}\right)+\left(b_{21}a_{12}+b_{22}a_{22}+b_{23}a_{32}\right)+\left(b_{31}a_{13}+b_{32}a_{23}+b_{33}a_{33}\right)$$

$$=\operatorname{tr}(BA)$$

9. $\left(A+A^T\right)^T=A^T+\left(A^T\right)^T=A^T+A=A+A^T$

10. a) $\left(A^2\right)^T=\left(AA\right)^T=A^TA^T=AA=A^2$ (since A is symmetric, $A^T=A$)

b) $\left(3A^2+2A-3I\right)^T=3\left(A^2\right)^T+2A^T-3I^T$

$$=3A^TA^T+2A^T-3I$$

$$=3A^2+2A-3I$$ (since A is symmetric, $A^T=A$)

11. a) $A=A^TA$

$A^T=\left(A^TA\right)^T=A^T\left(A^T\right)^T=A^TA=A$

b) $A=A^TA$

$=AA=A^2$ since by (a) A is symmetric

12. a) $W(x)=\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}$

b) $W(x)=\begin{bmatrix}e^{ax} & \ln(bx)\\ ae^{ax} & \dfrac{1}{x}\end{bmatrix}$

c) $W(x)=\begin{bmatrix}\sin 2x & \cos 3x\\ 2\cos 2x & -3\sin 3x\end{bmatrix}$

d) $W(x)=\begin{bmatrix}\frac{1}{x} & \frac{1}{x^2}\\ \frac{-1}{x^2} & \frac{-2}{x^3}\end{bmatrix}$

13. a) $A^2=\begin{bmatrix}\cos^4\theta+\sin^2\theta\cos^2\theta & \cos^3\theta\sin\theta+\sin^3\theta\cos\theta\\ \cos^3\theta\sin\theta+\sin^3\theta\cos\theta & \sin^2\theta\cos^2\theta+\sin^4\theta\end{bmatrix}$

$$=\begin{bmatrix}\cos^2\theta\left(\cos^2\theta+\sin^2\theta\right) & \cos\theta\sin\theta\left(\cos^2\theta+\sin^2\theta\right)\\ \cos\theta\sin\theta\left(\cos^2\theta+\sin^2\theta\right) & \sin^2\theta\left(\cos^2\theta+\sin^2\theta\right)\end{bmatrix}$$

$$=\begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{bmatrix}=A$$

$B^2=\left(I-A\right)^2=I^2-IA-AI+A^2=I-A-A+A=I-A=B$ (since $A^2=A$)

b) $AB=A\left(I-A\right)=AI-A^2=A-A=0$

$BA=\left(I-A\right)A=IA-A^2=A-A=0$

14. a) $\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ and $\begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix}$ b) $\begin{bmatrix}\pm 2 & 0\\ 0 & \pm 3\end{bmatrix}$ (4 different matrices).