MATHEMATICS 201-NYC-05
Vectors and Matrices
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Assignment #4
SOLUTIONS

This assignment is due ***Friday November 23*** at the beginning of the class. Complete solutions are expected.

Question 1 (12 points)

Prove that $W=\{(s,5,t):s,t\in\mathbb{R}\}$ with the following definitions for vector sum and scalar multiplication is a vector space. (Verify all 10 axioms!)

$$(u_1,5,u_3)\oplus(v_1,5,v_3)=(u_1+v_1,5,u_3+v_3+1)$$
$$k\odot(u_1,5,u_3)=(ku_1,5,k+ku_3-1)$$

Let $\vec{u}=(u_1,5,u_3)$, $\vec{v}=(v_1,5,v_3)$ and $\vec{w}=(w_1,5,w_3)$ be in W.

1. $\vec{u}\oplus\vec{v}=(u_1,5,u_3)\oplus(v_1,5,v_3)=(u_1+v_1,5,u_3+v_3+1)\in W$
2. $\vec{u}\oplus\vec{v}=(u_1+v_1,5,u_3+v_3+1)=(v_1+u_1,5,v_3+u_3+1)=\vec{v}\oplus\vec{u}$
3. $\vec{u}\oplus(\vec{v}\oplus\vec{w})=(u_1,5,u_3)\oplus(v_1+w_1,5,v_3+w_3+1)$
$$=(u_1+(v_1+w_1),5,u_3+(v_3+w_3+1)+1)$$
$$=((u_1+v_1)+w_1,5,(u_3+v_3+1)+w_3+1)$$
$$=(u_1+v_1,5,u_3+v_3+1)\oplus(w_1,5,w_3)$$
$$=(\vec{u}\oplus\vec{v})\oplus\vec{w}$$
4. $\vec{0}=0\odot\vec{u}=(0\cdot u_1,5,0+0\cdot u_3-1)=(0,5,-1)$
$\vec{u}\oplus\vec{0}=(u_1,5,u_3)\oplus(0,5,-1)=(u_1+0,5,u_3-1+1)=(u_1,5,u_3)=\vec{u}$
5. $-\vec{u}=(-1)\odot\vec{u}=((-1)\cdot u_1,5,-1+(-1)\cdot u_3-1)=(-u_1,5,-u_3-2)$
$\vec{u}+(-\vec{u})=(u_1,5,u_3)\oplus(-u_1,5,-u_3-2)=(u_1-u_1,5,u_3-u_3-2+1)=(0,5,-1)=\vec{0}$
6. $k\odot(u_1,5,u_3)=(ku_1,5,k+ku_3-1)\in W$
7. $k\odot(\vec{u}+\vec{v})=k\odot(u_1+v_1,5,u_3+v_3+1)$
$$=(k(u_1+v_1),5,k+k(u_3+v_3+1)-1)$$
$$=(ku_1+kv_1,5,(k+ku_3-1)+(k+kv_3-1)+1)$$
$$=(ku_1,5,k+ku_3-1)\oplus(kv_1,5,k+kv_3-1)$$
$$=(k\odot\vec{u})\oplus(k\odot\vec{v})$$

8. $(k+l)\odot\vec{u} = \left((k+l)u_1, 5, (k+l)+(k+l)u_3 - 1\right)$

$$= \left(ku_1 + lu_1, 5, (k+ku_3-1)+(l+lu_3-1)+1\right)$$
$$= \left(ku_1, 5, k+ku_3-1\right)\oplus\left(lu_1, 5, l+lu_3-1\right)$$
$$= \left(k\odot\vec{u}\right)\oplus\left(l\odot\vec{u}\right)$$

9. $k\odot\left(l\odot\vec{u}\right) = k\odot\left(lu_1, 5, l+lu_3-1\right)$

$$= \left(k\left(lu_3\right), 5, k+k\left(l+lu_3-1\right)-1\right)$$
$$= \left((kl)u_1, 5, kl+(kl)u_3-1\right) = (kl)\odot\vec{u}$$

10. $1\odot\vec{u} = 1\odot\left(u_1, 5, u_3\right) = \left(1\cdot u_1, 5, 1+1\cdot u_3 - 1\right) = \left(u_1, 5, u_3\right) = \vec{u}$

Question 2 (10 points)

For what values of t is the set $S = \left\{tx^2 + x + 2, x^2 + tx + 2, tx^2 + 3x + t\right\}$ linearly independent?

$$c_1\left(tx^2+x+2\right)+c_2\left(x^2+tx+2\right)+c_3\left(tx^2+3x+t\right)=0$$

$$\left[\begin{array}{ccc|c} t & 1 & t & 0 \\ 1 & t & 3 & 0 \\ 2 & 2 & t & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 2 & 2 & t & 0 \\ 1 & t & 3 & 0 \\ t & 1 & t & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to 2R_3 - tR_1]{R_2 \to 2R_2 - R_1} \left[\begin{array}{ccc|c} 2 & 2 & t & 0 \\ 0 & 2t-2 & 6-t & 0 \\ 0 & 2-2t & 2t-t^2 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{ccc|c} 2 & 2 & t & 0 \\ 0 & 2t-2 & 6-t & 0 \\ 0 & 0 & -t^2+t+6 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to \frac{1}{-t^2+t+6}R_3]{R_1 \to \frac{1}{2}R_1,\ R_2 \to \frac{1}{2t-2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2}t & 0 \\ 0 & 1 & \frac{6-t}{2t-2} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \qquad \begin{array}{l} c_3 = 0 \\ c_2 = 0 \\ c_1 = 0 \end{array}$$

Assuming $2t-2 \neq 0$ and $-t^2+t+6=0$

$t \neq 1$ $\qquad -(t-3)(t+2)=0$

$t = -2, 3$

If $t = 1$

then the augmented matrix becomes

$$\left[\begin{array}{ccc|c} 2 & 2 & 1 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_2 \to \frac{1}{5}R_2]{R_1 \to \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} c_3 = 0 \\ c_2 = t \\ c_1 = -t \end{array}$$

so S is linearly dependent

If $t=-2$

then the augmented matrix for $t=-2$ becomes

$$\left[\begin{array}{ccc|c} 2 & 2 & -2 & 0 \\ 0 & -6 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_2 \to \frac{-1}{6} R_2]{R_1 \to \frac{1}{2} R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & \frac{-4}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and the solution is $c_3 = t,\ c_2 = \frac{4}{3}t,\ c_1 = -\frac{1}{3}t$, so S is linearly dependent

for $t=3$ the augmented matrix for becomes

$$\left[\begin{array}{ccc|c} 2 & 2 & 3 & 0 \\ 0 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_2 \to \frac{1}{4} R_2]{R_1 \to \frac{1}{2} R_1} \left[\begin{array}{ccc|c} 1 & 1 & \frac{3}{2} & 0 \\ 0 & 1 & \frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and the solution is $c_3 = t,\ c_2 = \frac{-3}{4}t,\ c_1 = \frac{-3}{4}t$, so S is linearly dependent

Ergo, if $t \neq -2,1,3$ then S is linearly independent.

Question 3 (20 points)

Let $W = \{p(x) : p(x) \in P_3,\ p(3) = 0\}$ and $U = \{p(x) : p(x) \in P_3,\ p(2) = 0,\ p(-2) = 0\}$.

a) Show that W and U are subspaces of P_3.

For W: $p(x) = x - 3 \in W$ since $p(3) = 0$ and $p(x) \in P_3$, thus W is nonempty.

Let $p(x) \in W$ and $q(x) \in W$. Then $p(3) = 0$ and $q(3) = 0$.

1. $p(x) + q(x) \in W$ since $p(3) + q(3) = 0 + 0 = 0$.
2. $kp(x) \in$ since $kp(3) = k0 = 0$.

Thus W is a subspace or P_3.

For U: $p(x) = x^2 - 4 \in U$ since $p(2) = 0$, $p(-2) = 0$ and $p(x) \in P_2$, thus U is nonempty.

Let $p(x) \in U$ and $q(x) \in U$. Then $p(2) = 0$, $p(-2) = 0$, $q(2) = 0$, $q(-2) = 0$.

1. $p(x) + q(x) \in U$ since $p(2) + q(2) = 0 + 0 = 0$
   $p(-2) + q(-2) = 0 + 0 = 0$
2. $kp(x) \in$ since $kp(2) = k0 = 0$ and $kp(-2) = k0 = 0$.

Thus U is a subspace or P_3.

b) Find a basis and the dimension for W.

Let $p(x) = a + bx + cx^2 + dx^3 \in W$

Then $p(3) = a + 3b + 9c + 27d = 0$

Let $d = t$, $c = s$ and $b = r$, then $a = -3r - 9s - 27t$

Thus $p(x) = (-3r - 9s - 27t) + rx + sx^2 + tx^3$.

$$= r(-3 + x) + s(-9 + x^2) + t(-27 + x^3)$$

Hence if $B_W = \{-3 + x, -9 + x^2, -27 + x^3\}$, then $W = span(B_w)$.

Verifying independence:

$$B_W = c_1(-3 + x) + c_2(-9 + x^2) + c_3(-27 + x^3)$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -3 & -9 & -27 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & -9 & -27 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_4 + 3R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -9 & -27 & 0 \end{array}\right] \xrightarrow{R_4 \to R_4 + 9R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -27 & 0 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4 + 27R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} c_3 = 0 \\ c_2 = 0 \\ c_1 = 0 \end{array}$$

Hence B_w is linearly independent, thus B_W is a basis for *W* and $\dim(W) = 3$.

c) Find a basis and the dimension for *U*.

Let $p(x) = a + bx + cx^2 + dx^3 \in W$.

We have $p(2) = a + 2b + 4c + 8d = 0$

$p(-2) = a - 2b + 4c - 8d = 0$

$$\left[\begin{array}{cccc|c} 1 & 2 & 4 & 8 & 0 \\ 1 & -2 & 4 & -8 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 2 & 4 & 8 & 0 \\ 0 & -4 & 0 & -16 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to \frac{1}{6} R_2} \left[\begin{array}{cccc|c} 1 & 2 & 4 & 8 & 0 \\ 0 & 1 & 0 & 4 & 0 \end{array}\right]$$

Let $d = t$, $c = s$, then $b = -4t$ and $a = -4s$

Thus $p(x) = -4s - 4tx + sx^2 + tx^3 = s(-4 + x^2) + t(-4x + x^3)$.

Hence if $B_U = \{-4 + x^2, -4x + x^3\}$, then $U = span(B_U)$ and since B_U is linearly independent (two nonparallel vectors), B_U is a basis for *W* and $\dim(U) = 2$.

d) Is $q(x)=2x^3-6x^2-8x+24$ an element of W? If yes, find the coordinates of $q(x)$ relative to the basis found in (b).

$$c_1(-3+x)+c_2(-9+x^2)+c_3(-27+x^3)=2x^3-6x^2-8x+24$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -6 \\ 1 & 0 & 0 & -8 \\ -3 & -9 & -27 & 24 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 2 \\ -3 & -9 & -27 & 24 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4+3R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 2 \\ 0 & -9 & -27 & 0 \end{array}\right] \xrightarrow{R_4 \to R_4+9R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -27 & -54 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4+27R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} c_3=2 \\ c_2=-6 \\ c_1=-8 \end{array}$$

Hence $q(x)\in W$ and $(q(x))_{B_W}=(-8,-6,2)$

e) Find a basis for $W\cap U$ and determine the dimension of $W\cap U$.

$$p(3)=a+3b+9c+27d=0$$
$$p(2)=a+2b+4c+8d=0$$
$$p(-2)=a-2b+4c-8d=0$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 9 & 27 & 0 \\ 1 & 2 & 4 & 8 & 0 \\ 1 & -2 & 4 & -8 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3-R_1]{R_2 \to R_2-R_1} \left[\begin{array}{cccc|c} 1 & 3 & 9 & 27 & 0 \\ 0 & -1 & -5 & -19 & 0 \\ 0 & -5 & -5 & -35 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3-5R_2} \left[\begin{array}{cccc|c} 1 & 3 & 9 & 27 & 0 \\ 0 & -1 & -5 & -19 & 0 \\ 0 & 0 & 20 & 60 & 0 \end{array}\right] \xrightarrow[R_3 \to \frac{1}{20}R_3]{R_2 \to -R_2} \left[\begin{array}{cccc|c} 1 & 3 & 9 & 27 & 0 \\ 0 & 1 & 5 & 19 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{array}\right]$$

$$d=t$$
$$c=-3t$$
$$b=-4t$$
$$a=12t$$

Thus $p(x)=12t-4tx-3tx^2+tx^3=t(12-4x-3x^2+x^3)$.

Hence if $B_{W\cap U}=\{x^3-3x^2+4x-12\}$, then $W\cap U=span(B_{W\cap U})$ and since $B_{W\cap U}$ B_U is linearly independent (only one vector), $B_{W\cap U}$ is a basis for $W\cap U$ and $\dim(W\cap U)=1$.

f) Is $q(x)=2x^3-6x^2-8x+24$ an element of $W\cap U$? If yes, find the coordinates of $q(x)$ relative to the basis found in (e).

$q(x)=2x^3-6x^2-8x+24=2(x^3-3x^2+4x-12)$

Hence $q(x)\in W\cap U$ and $q(x)_{B_{W\cap U}}=(2)$

Question 4 (8 points)

Let $\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ be an orthonormal basis for $\mathbb{R}^3$.

a) Show that if $\vec{w}\in\mathbb{R}^3$, then $\vec{w}$ can be written as $\vec{w}=(\vec{w}\bullet\vec{v}_1)\vec{v}_1+(\vec{w}\bullet\vec{v}_2)\vec{v}_2+(\vec{w}\bullet\vec{v}_3)\vec{v}_3$.

Since $\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ is a basis for $\mathbb{R}^3$, then

$$\vec{w}=c_1\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3$$

We have
$$\begin{aligned}\vec{w}\bullet\vec{v}_1&=(c_1\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3)\bullet\vec{v}_1\\&=c_1\vec{v}_1\bullet\vec{v}_1+c_2\vec{v}_2\bullet\vec{v}_1+c_3\vec{v}_3\bullet\vec{v}_1\\&=c_1(1)+c_2(0)+c_3(0) &&\text{Since the basis is orthonormal}\\&=c_1\end{aligned}$$

$$\begin{aligned}\vec{w}\bullet\vec{v}_2&=(c_1\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3)\bullet\vec{v}_2\\&=c_1\vec{v}_1\bullet\vec{v}_2+c_2\vec{v}_2\bullet\vec{v}_2+c_3\vec{v}_3\bullet\vec{v}_2\\&=c_1(0)+c_2(1)+c_3(0) &&\text{Since the basis is orthonormal}\\&=c_2\end{aligned}$$

$$\begin{aligned}\vec{w}\bullet\vec{v}_3&=(c_1\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3)\bullet\vec{v}_3\\&=c_1\vec{v}_1\bullet\vec{v}_3+c_2\vec{v}_2\bullet\vec{v}_3+c_3\vec{v}_3\bullet\vec{v}_3\\&=c_1(0)+c_2(0)+c_3(1) &&\text{Since the basis is orthonormal}\\&=c_3\end{aligned}$$

Thus $\vec{w}=(\vec{w}\bullet\vec{v}_1)\vec{v}_1+(\vec{w}\bullet\vec{v}_2)\vec{v}_2+(\vec{w}\bullet\vec{v}_3)\vec{v}_3$.

b) Show that if $\vec{w}\in\mathbb{R}^3$, then $\|\vec{w}\|=(\vec{w}\bullet\vec{v}_1)^2+(\vec{w}\bullet\vec{v}_2)^2+(\vec{w}\bullet\vec{v}_3)^2$.

$$\begin{aligned}LS&=\|\vec{w}\|^2=\vec{w}\bullet\vec{w}\\&=((\vec{w}\bullet\vec{v}_1)\vec{v}_1+(\vec{w}\bullet\vec{v}_2)\vec{v}_2+(\vec{w}\bullet\vec{v}_3)\vec{v}_3)\bullet\vec{w}\\&=(\vec{w}\bullet\vec{v}_1)(\vec{v}_1\bullet\vec{w})+(\vec{w}\bullet\vec{v}_2)(\vec{v}_2\bullet\vec{w})+(\vec{w}\bullet\vec{v}_3)(\vec{v}_3\bullet\vec{w})\\&=(\vec{w}\bullet\vec{v}_1)^2+(\vec{w}\bullet\vec{v}_2)^2+(\vec{w}\bullet\vec{v}_3)^2\\&=RS\end{aligned}$$