MATHEMATICS 201-NYC-05
Vectors and Matrices
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Assignment #3
SOLUTIONS

Question 1 (7 points)

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors. Let $\vec{w} = k\vec{u} + l\vec{v}$ where $k = \|\vec{v}\|$ and $l = \|\vec{u}\|$. Show that $\vec{w}$ bisects the angle between $\vec{u}$ and $\vec{v}$. That is, if θ is the angle between $\vec{u}$ and $\vec{v}$, then the angle between $\vec{u}$ and $\vec{w}$ is $\frac{\theta}{2}$, and the angle between $\vec{v}$ and $\vec{w}$ is $\frac{\theta}{2}$. *Hint*: Use the trigonometric identity $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$.

Let ϕ be the angle between $\vec{u}$ and $\vec{w}$, then $\cos\phi = \dfrac{\vec{u}\bullet\vec{w}}{\|\vec{u}\|\|\vec{w}\|}$

We have that

$$\begin{aligned}\vec{u}\bullet\vec{w} &= \vec{u}\bullet\left(k\vec{u} + l\vec{v}\right)\\ &= k\vec{u}\bullet\vec{u} + l\vec{u}\bullet\vec{v}\\ &= \|\vec{v}\|\|\vec{u}\|^2 + \|\vec{u}\|\|\vec{u}\|\|\vec{v}\|\cos\theta\\ &= \|\vec{u}\|^2\|\vec{v}\|\left(1+\cos\theta\right)\end{aligned}$$

$$\begin{aligned}\|\vec{w}\| &= \sqrt{\vec{w}\bullet\vec{w}}\\ &= \sqrt{\left(k\vec{u} + l\vec{v}\right)\bullet\left(k\vec{u} + l\vec{v}\right)}\\ &= \sqrt{k^2\vec{u}\bullet\vec{u} + 2kl\vec{u}\bullet\vec{v} + l^2\vec{v}\bullet\vec{v}}\\ &= \sqrt{\|\vec{v}\|^2\|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\|\|\vec{u}\|\|\vec{v}\|\cos\theta + \|\vec{u}\|^2\|\vec{v}\|^2}\\ &= \sqrt{\|\vec{u}\|^2\|\vec{v}\|^2\left(2+2\cos\theta\right)}\\ &= \|\vec{u}\|\|\vec{v}\|\sqrt{2}\sqrt{1+\cos\theta}\end{aligned}$$

Hence

$$\begin{aligned}\cos\phi &= \frac{\vec{u}\bullet\vec{w}}{\|\vec{u}\|\|\vec{w}\|} = \frac{\|\vec{u}\|^2\|\vec{v}\|\left(1+\cos\theta\right)}{\|\vec{u}\|\|\vec{u}\|\|\vec{v}\|\sqrt{2}\sqrt{1+\cos\theta}}\\ &= \frac{1}{\sqrt{2}}\sqrt{1+\cos\theta}\\ &= \frac{1}{\sqrt{2}}\sqrt{2\cos^2\tfrac{\theta}{2}}\\ &= \cos\tfrac{\theta}{2}\end{aligned}$$

Since $0 \le \phi, \theta \le \pi$, then $\phi = \frac{\theta}{2}$

Let Ψ be the angle between $\vec{v}$ and $\vec{w}$, then

$$\cos\Psi = \frac{\vec{v}\bullet\vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \frac{\|\vec{u}\|\|\vec{v}\|^2\left(1+\cos\theta\right)}{\|\vec{v}\|\|\vec{u}\|\|\vec{v}\|\sqrt{2}\sqrt{1+\cos\theta}} = \frac{1}{\sqrt{2}}\sqrt{1+\cos\theta} = \cos\tfrac{\theta}{2}$$

Ergo, $\Psi = \frac{\theta}{2}$

Thus $\vec{w}$ bisects the angle between $\vec{u}$ and $\vec{v}$.

Question 2 (20 points)

For a given set of vectors $\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ in $\mathbb{R}^3$. We define the vectors $\vec{w}_1$, $\vec{w}_2$ and $\vec{w}_3$ as

$$\vec{w}_1=\vec{v}_1$$
$$\vec{w}_2=\vec{v}_2-\frac{\vec{v}_2\bullet\vec{w}_1}{\vec{w}_1\bullet\vec{w}_1}\vec{w}_1$$
$$\vec{w}_3=\vec{v}_3-\frac{\vec{v}_3\bullet\vec{w}_1}{\vec{w}_1\bullet\vec{w}_1}\vec{w}_1-\frac{\vec{v}_3\bullet\vec{w}_2}{\vec{w}_2\bullet\vec{w}_2}\vec{w}_2$$

and the vectors $\vec{u}_1$, $\vec{u}_2$ and $\vec{u}_3$ by

$$\vec{u}_1=\frac{\vec{w}_1}{\|\vec{w}_1\|} \qquad \vec{u}_2=\frac{\vec{w}_2}{\|\vec{w}_2\|} \qquad \vec{u}_3=\frac{\vec{w}_3}{\|\vec{w}_3\|}$$

This is known as the ***Gram-Schmidt Orthonormalization Process***.

a) For the vectors $\vec{v}_1=(1,0,1)$, $\vec{v}_2=(1,1,2)$ and $\vec{v}_3=(1,2,1)$, find $\vec{w}_1$, $\vec{w}_2$, $\vec{w}_3$, $\vec{u}_1$, $\vec{u}_2$ and $\vec{u}_3$.

$$\vec{w}_1=\vec{v}_1=(1,0,1)$$
$$\vec{w}_2=\vec{v}_2-\frac{\vec{v}_2\bullet\vec{w}_1}{\vec{w}_1\bullet\vec{w}_1}\vec{w}_1=(1,1,2)-\frac{3}{2}(1,0,1)=\left(\frac{-1}{2},1,\frac{1}{2}\right)$$
$$\vec{w}_3=\vec{v}_3-\frac{\vec{v}_3\bullet\vec{w}_1}{\vec{w}_1\bullet\vec{w}_1}\vec{w}_1-\frac{\vec{v}_3\bullet\vec{w}_2}{\vec{w}_2\bullet\vec{w}_2}\vec{w}_2=(1,2,1)-\frac{2}{2}(1,0,1)-\frac{2}{\frac{3}{2}}\left(\tfrac{-1}{2},1,\tfrac{1}{2}\right)=\left(\frac{2}{3},\frac{2}{3},\frac{-2}{3}\right)$$
$$\vec{u}_1=\frac{\vec{w}_1}{\|\vec{w}_1\|}=\frac{1}{\sqrt{2}}(1,0,1)=\left(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2}\right)$$
$$\vec{u}_2=\frac{\vec{w}_2}{\|\vec{w}_2\|}=\frac{\sqrt{2}}{\sqrt{3}}\left(\frac{-1}{2},1,\frac{1}{2}\right)=\left(\frac{-\sqrt{6}}{6},\frac{\sqrt{6}}{3},\frac{\sqrt{6}}{6}\right)$$
$$\vec{u}_3=\frac{\vec{w}_3}{\|\vec{w}_3\|}=\frac{\sqrt{3}}{2}\left(\frac{2}{3},\frac{2}{3},\frac{-2}{3}\right)=\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{-\sqrt{3}}{3}\right)$$

b) Show that the set $\{\vec{u}_1,\vec{u}_2,\vec{u}_3\}$ found in (a) is orthonormal.

$$\vec{u}_1\bullet\vec{u}_2=\left(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2}\right)\bullet\left(\frac{-\sqrt{6}}{6},\frac{\sqrt{6}}{3},\frac{\sqrt{6}}{6}\right)=-\frac{\sqrt{12}}{12}+\frac{\sqrt{12}}{12}=0$$
$$\vec{u}_1\bullet\vec{u}_3=\left(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2}\right)\bullet\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{-\sqrt{3}}{3}\right)=\frac{\sqrt{6}}{6}-\frac{\sqrt{6}}{6}=0$$
$$\vec{u}_2\bullet\vec{u}_3=\left(\frac{-\sqrt{6}}{6},\frac{\sqrt{6}}{3},\frac{\sqrt{6}}{6}\right)\bullet\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{-\sqrt{3}}{3}\right)=\frac{-\sqrt{18}}{18}+\frac{\sqrt{18}}{9}-\frac{\sqrt{18}}{18}=0$$
$$\|\vec{u}_1\|=\sqrt{\left(\tfrac{\sqrt{2}}{2}\right)^2+0+\left(\tfrac{\sqrt{2}}{2}\right)^2}=\sqrt{\tfrac{1}{2}+\tfrac{1}{2}}=1$$
$$\|\vec{u}_2\|=\sqrt{\left(\tfrac{-\sqrt{6}}{6}\right)^2+\left(\tfrac{\sqrt{6}}{3}\right)^2+\left(\tfrac{\sqrt{6}}{6}\right)^2}=\sqrt{\tfrac{1}{6}+\tfrac{2}{3}+\tfrac{1}{6}}=1$$

$$\|\vec{u}_3\| = \sqrt{\left(\tfrac{\sqrt{3}}{3}\right)^2 + \left(\tfrac{\sqrt{3}}{3}\right)^2 + \left(\tfrac{-\sqrt{3}}{3}\right)^2} = \sqrt{\tfrac{1}{3}+\tfrac{1}{3}+\tfrac{1}{3}} = 1$$

c) Prove that for <u>any</u> nonzero vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ such that $\vec{w}_1$, $\vec{w}_2$ and $\vec{w}_3$ are all nonzero (this is equivalent to saying that the vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are linearly independent), then

i) $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is an orthogonal set

$$\vec{w}_1 \bullet \vec{w}_2 = \vec{w}_1 \bullet \left(\vec{v}_2 - \frac{\vec{v}_2 \bullet \vec{w}_1}{\vec{w}_1 \bullet \vec{w}_1}\vec{w}_1\right) = \vec{w}_1 \bullet \vec{v}_2 - \frac{\vec{v}_2 \bullet \vec{w}_1}{\cancel{\vec{w}_1 \bullet \vec{w}_1}}\cancel{\vec{w}_1 \bullet \vec{w}_1} = \vec{w}_1 \bullet \vec{v}_2 - \vec{w}_1 \bullet \vec{v}_2 = 0$$

$$\begin{aligned}\vec{w}_1 \bullet \vec{w}_3 &= \vec{w}_1 \bullet \left(\vec{v}_3 - \frac{\vec{v}_3 \bullet \vec{w}_1}{\vec{w}_1 \bullet \vec{w}_1}\vec{w}_1 - \frac{\vec{v}_3 \bullet \vec{w}_2}{\vec{w}_2 \bullet \vec{w}_2}\vec{w}_2\right) \\ &= \vec{w}_1 \bullet \vec{v}_3 - \frac{\vec{v}_3 \bullet \vec{w}_1}{\cancel{\vec{w}_1 \bullet \vec{w}_1}}\cancel{\vec{w}_1 \bullet \vec{w}_1} - \frac{\vec{v}_3 \bullet \vec{w}_2}{\vec{w}_2 \bullet \vec{w}_2}\vec{w}_1 \bullet \vec{w}_2 \\ &= \vec{w}_1 \bullet \vec{v}_3 - \vec{w}_1 \bullet \vec{v}_3 - 0 \qquad \text{since } \vec{w}_1 \bullet \vec{w}_2 = 0 \\ &= 0\end{aligned}$$

$$\begin{aligned}\vec{w}_2 \bullet \vec{w}_3 &= \vec{w}_2 \bullet \left(\vec{v}_3 - \frac{\vec{v}_3 \bullet \vec{w}_1}{\vec{w}_1 \bullet \vec{w}_1}\vec{w}_1 - \frac{\vec{v}_3 \bullet \vec{w}_2}{\vec{w}_2 \bullet \vec{w}_2}\vec{w}_2\right) \\ &= \vec{w}_2 \bullet \vec{v}_3 - \frac{\vec{v}_3 \bullet \vec{w}_1}{\vec{w}_1 \bullet \vec{w}_1}\vec{w}_2 \bullet \vec{w}_1 - \frac{\vec{v}_3 \bullet \vec{w}_2}{\cancel{\vec{w}_2 \bullet \vec{w}_2}}\cancel{\vec{w}_2 \bullet \vec{w}_2} \\ &= \vec{w}_2 \bullet \vec{v}_3 - 0 - \vec{w}_2 \bullet \vec{v}_3 \qquad \text{since } \vec{w}_2 \bullet \vec{w}_1 = 0 \\ &= 0\end{aligned}$$

ii) $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthonormal set

$$\vec{u}_1 \bullet \vec{u}_2 = \frac{\vec{w}_1}{\|\vec{w}_1\|} \bullet \frac{\vec{w}_2}{\|\vec{w}_2\|} = \frac{\vec{w}_1 \bullet \vec{w}_2}{\|\vec{w}_1\|\|\vec{w}_2\|} = 0 \qquad \text{since } \vec{w}_1 \bullet \vec{w}_2 = 0 \text{ from (i)}$$

$$\vec{u}_1 \bullet \vec{u}_3 = \frac{\vec{w}_1}{\|\vec{w}_1\|} \bullet \frac{\vec{w}_3}{\|\vec{w}_3\|} = \frac{\vec{w}_1 \bullet \vec{w}_3}{\|\vec{w}_1\|\|\vec{w}_3\|} = 0 \qquad \text{since } \vec{w}_1 \bullet \vec{w}_3 = 0 \text{ from (i)}$$

$$\vec{u}_2 \bullet \vec{u}_3 = \frac{\vec{w}_2}{\|\vec{w}_2\|} \bullet \frac{\vec{w}_3}{\|\vec{w}_3\|} = \frac{\vec{w}_2 \bullet \vec{w}_3}{\|\vec{w}_2\|\|\vec{w}_3\|} = 0 \qquad \text{since } \vec{w}_2 \bullet \vec{w}_3 = 0 \text{ from (i)}$$

$$\|\vec{u}_1\| = \left\|\frac{\vec{w}_1}{\|\vec{w}_1\|}\right\| = \frac{1}{\|\vec{w}_1\|}\|\vec{w}_1\| = 1$$

$$\|\vec{u}_2\| = \left\|\frac{\vec{w}_2}{\|\vec{w}_2\|}\right\| = \frac{1}{\|\vec{w}_2\|}\|\vec{w}_2\| = 1$$

$$\|\vec{u}_3\| = \left\|\frac{\vec{w}_3}{\|\vec{w}_3\|}\right\| = \frac{1}{\|\vec{w}_3\|}\|\vec{w}_3\| = 1$$

d) Verify your answer in (a) with Maple. That is, find the vectors $\vec{w}_1$, $\vec{w}_2$, $\vec{w}_3$, $\vec{u}_1$, $\vec{u}_2$ and $\vec{u}_3$ with Maple.

e) Let Q be the matrix in which has for rows the vector $\vec{u}_1$, $\vec{u}_2$ and $\vec{u}_3$. Find the inverse of Q with Maple. What is the relationship between Q and its inverse?

Question 3 (7 points)

Consider the points $A(2,-1,3)$, $B(6,3,-3)$ and $C(6,-7,-4)$. Using Maple,

a) find the equation of the line l passing through the points A and B;
b) plot the line l found in (a) along with the direction vector for the line (choose a view such that both objects, and the relationship between them are clearly seen);
c) find the distance between the point C and the line l found in (a);
d) find the point Q on l closest to the point C.
e) find the distance between the line l found in (a) and the line $l_2 : \frac{x-1}{2} = \frac{y+1}{-3} = \frac{z-1}{4}$.

Question 4 (9 points)

Consider the four points $A(2,-1,3)$, $B(3,1,1)$, $C(-2,-2,1)$ and $P(14,-6,-6)$. Using Maple,

a) find the equation of the plane π passing through the points A, B and C (in general form);
b) find the equation of the line l passing through P and perpendicular to the plane π;
c) find the point Q on π closest to the point P;
d) find the distance between the point P and the plane π;
e) plot the line l, the plane π and the normal vector (choose a view such that all three objects, and the relationships between them, are clearly seen);
f) find the volume of the tetrahedron $ABCP$.

Question 5 (7 points)

Consider the planes $\pi_1 : x-3y+z=8$ and $\pi_2 : 5x-2y+3z=5$. Using Maple,

a) find the intersection of the planes π_1 and π_2 (Expressed in vector form) ;
b) find the angle between the two planes (answer should be in degrees);
c) find the equation for a plane π_3 that is not parallel to either π_1 and π_2 such that $\pi_1 \cap \pi_2 \cap \pi_3 = \varnothing$; (Verify your answer!)
d) plot the three planes (choose a view that shows why the planes do not all intersect at a point).