

MATHEMATICS 201-NYC-05

Vectors and Matrices

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Assignment #2 SOLUTIONS

This assignment is due **Tuesday October 2** at the beginning of the class.

For questions involving Maple, a print-out of your work is expected, where your name is written in the Worksheet, and each question is clearly labeled.

Question 1 (26 points)

Consider the matrix $A = \begin{bmatrix} 2 & 2 & 0 \\ -1 & 2 & 3 \\ 1 & 2 & -1 \end{bmatrix}$.

a) Find the eigenvalues and eigenvectors of A .

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda - 2 & -2 & 0 \\ 1 & \lambda - 2 & -3 \\ -1 & -2 & \lambda + 1 \end{vmatrix} \\ &= (\lambda - 2) \begin{vmatrix} \lambda - 2 & -3 \\ -2 & \lambda + 1 \end{vmatrix} + 2 \begin{vmatrix} 1 & -3 \\ -1 & \lambda + 1 \end{vmatrix} \\ &= (\lambda - 2)((\lambda - 2)(\lambda + 1) - 6) + 2((\lambda + 1) - 3) \\ &= (\lambda - 2)(\lambda^2 - \lambda - 8) + 2(\lambda - 2) \\ &= (\lambda - 2)(\lambda^2 - \lambda - 6) \\ &= (\lambda - 2)(\lambda + 2)(\lambda - 3) \end{aligned}$$

Hence the eigenvalues are $\lambda = -2, 2, 3$

For $\lambda_1 = -2$, $(-2I - A)X = 0$ yields the augmented matrix

$$\begin{aligned} &\begin{bmatrix} -4 & -2 & 0 & | & 0 \\ 1 & -4 & -3 & | & 0 \\ -1 & -2 & -1 & | & 0 \end{bmatrix} \begin{array}{l} R_2 \rightarrow 4R_2 + R_1 \\ R_3 \rightarrow 4R_2 - R_1 \end{array} \begin{bmatrix} -4 & -2 & 0 & | & 0 \\ 0 & -18 & -12 & | & 0 \\ 0 & -6 & -4 & | & 0 \end{bmatrix} \\ &\quad \begin{array}{l} R_3 \rightarrow 3R_3 - R_2 \\ R_1 \rightarrow \frac{-1}{4}R_1 \\ R_2 \rightarrow \frac{-1}{18}R_2 \end{array} \begin{bmatrix} -4 & -2 & 0 & | & 0 \\ 0 & -18 & -12 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 & | & 0 \\ 0 & 1 & \frac{2}{3} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \end{aligned}$$

So the solution is $(\frac{1}{3}t, \frac{-2}{3}t, t) = \frac{1}{3}t(1, -2, 3)$.

Hence the eigenvectors for $\lambda_1 = -2$ are the multiples of $(1, -2, 3)$.

For $\lambda_2 = 2$, $(2I - A)X = 0$ yields the augmented matrix

$$\begin{aligned} & \left[\begin{array}{ccc|c} 0 & -2 & 0 & 0 \\ 1 & 0 & -3 & 0 \\ -1 & -2 & 3 & 0 \end{array} \right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & -2 & 0 & 0 \\ -1 & -2 & 3 & 0 \end{array} \right] \\ & \xrightarrow{R_3 \rightarrow R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \\ & \xrightarrow{R_2 \rightarrow \frac{-1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

So the solution is $(3t, 0, t) = t(3, 0, 1)$.

Hence the eigenvectors for $\lambda_2 = 2$ are the multiples of $(3, 0, 1)$.

For $\lambda_3 = 3$, $(2I - A)X = 0$ yields the augmented matrix

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 1 & 1 & -3 & 0 \\ -1 & -2 & 4 & 0 \end{array} \right] \xrightarrow{\substack{R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 + R_1}} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -4 & 4 & 0 \end{array} \right] \\ & \xrightarrow{R_3 \rightarrow 3R_3 + 4R_1} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R_2 \rightarrow \frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

So the solution is $(2t, t, t) = t(2, 1, 1)$.

Hence the eigenvectors for $\lambda_3 = 3$ are the multiples of $(2, 1, 1)$.

b) Verify that $\text{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3$.

$$\text{tr}(A) = 2 + 2 - 1 = 3$$

$$\lambda_1 + \lambda_2 + \lambda_3 = -2 + 2 + 3 = 3$$

$$\therefore \text{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3$$

c) Verify that $\det(A) = \lambda_1 \lambda_2 \lambda_3$.

$$\det(A) = 2 \begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} - 2 \begin{vmatrix} -1 & 3 \\ 1 & -1 \end{vmatrix} = 2(-8) - 2(-2) = -12$$

$$\lambda_1 \lambda_2 \lambda_3 = (-2)(2)(3) = -12$$

$$\therefore \det(A) = \lambda_1 \lambda_2 \lambda_3$$

d) Consider the matrix P whose columns are the eigenvectors of A found in (a). Find P .

$$P = \begin{bmatrix} 1 & 3 & 2 \\ -2 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

e) Find P^{-1} .

$$\begin{aligned} & \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 + 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 6 & 5 & 2 & 1 & 0 \\ 0 & -8 & -5 & -3 & 0 & 1 \end{array} \right] \\ & \begin{array}{l} R_3 \rightarrow 3R_3 + 4R_2 \\ R_1 \rightarrow 5R_1 - 2R_3 \\ R_2 \rightarrow R_2 - R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 6 & 5 & 2 & 1 & 0 \\ 0 & 0 & 5 & -1 & 4 & 3 \end{array} \right] \left[\begin{array}{ccc|ccc} 5 & 15 & 0 & 7 & -8 & -6 \\ 0 & 6 & 0 & 3 & -3 & -3 \\ 0 & 0 & 5 & -1 & 4 & 3 \end{array} \right] \\ & \begin{array}{l} R_1 \rightarrow \frac{1}{10}R_1 \\ R_2 \rightarrow \frac{1}{6}R_2 \\ R_3 \rightarrow \frac{1}{5}R_3 \end{array} \left[\begin{array}{ccc|ccc} 10 & 0 & 0 & -1 & -1 & 3 \\ 0 & 6 & 0 & 3 & -3 & -3 \\ 0 & 0 & 5 & -1 & 4 & 3 \end{array} \right] \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-1}{10} & \frac{-1}{10} & \frac{3}{10} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \\ 0 & 0 & 1 & \frac{-1}{5} & \frac{4}{5} & \frac{3}{5} \end{array} \right] \\ & \text{Thus } P^{-1} = \begin{bmatrix} \frac{-1}{10} & \frac{-1}{10} & \frac{3}{10} \\ \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \\ \frac{-1}{5} & \frac{4}{5} & \frac{3}{5} \end{bmatrix} \end{aligned}$$

f) Use (d) and (e) to find the matrix D defined as $D = P^{-1}AP$. What do the entries in D correspond to? (What was done in (a), (d), (e) and (f) is referred to as the diagonalization of A)

$$\begin{aligned} D = P^{-1}AP &= \begin{bmatrix} \frac{-1}{10} & \frac{-1}{10} & \frac{3}{10} \\ \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \\ \frac{-1}{5} & \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ -1 & 2 & 3 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ -2 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{-3}{5} \\ 1 & -1 & -1 \\ \frac{-3}{5} & \frac{12}{5} & \frac{9}{5} \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ -2 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \end{aligned}$$

The entries on the main diagonal of D correspond to the eigenvalues of A .

g) Show that if P is an invertible matrix such that $D = P^{-1}AP$, then $A = PDP^{-1}$.

$$D = P^{-1}AP$$

$$PDP^{-1} = PP^{-1}APP^{-1}$$

$$PDP^{-1} = IAI$$

$$PDP^{-1} = A$$

$$A = PDP^{-1}$$

- h) If P is an invertible matrix such that $A = PDP^{-1}$ and k is a positive integer, then $A^k = PD^kP^{-1}$. Prove this result for $k = 3$, that is, prove that $A^3 = PD^3P^{-1}$.

$$\begin{aligned} A^3 &= (PDP^{-1})^3 \\ &= PDP^{-1}PDP^{-1}PDP^{-1} \\ &= PDIDIDP^{-1} \\ &= PD^3P^{-1} \end{aligned}$$

- i) Use the result from (h) to find A^5 (This is a procedure that enables a relatively quick calculation of powers of matrices when the exponent is large).

$$\begin{aligned} A^5 &= PD^5P^{-1} \\ &= \begin{bmatrix} 1 & 3 & 2 \\ -2 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^5 \begin{bmatrix} -\frac{1}{10} & -\frac{1}{10} & \frac{3}{10} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{5} & \frac{4}{5} & \frac{3}{5} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 & 2 \\ -2 & 0 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 32 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 243 \end{bmatrix} \begin{bmatrix} -\frac{1}{10} & -\frac{1}{10} & \frac{3}{10} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{5} & \frac{4}{5} & \frac{3}{5} \end{bmatrix} \\ &= \begin{bmatrix} 32 & 96 & 486 \\ -64 & 0 & 243 \\ 96 & 32 & 243 \end{bmatrix} \begin{bmatrix} -\frac{1}{10} & -\frac{1}{10} & \frac{3}{10} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{5} & \frac{4}{5} & \frac{3}{5} \end{bmatrix} \\ &= \begin{bmatrix} -46 & 344 & 234 \\ -55 & 188 & 165 \\ -23 & 188 & 101 \end{bmatrix} \end{aligned}$$

- j) Verify your answers in (a), (b), (c), (e), (f) and (i) using Maple.

[See Maple](#)

Question 2 (4 points)

Use Maple to find the lowest degree polynomial (if any) passing through the points $(-2, 67)$, $(-1, 4)$, $(1, -2)$, $(3, 112)$ and $(5, 970)$. Your solution should be clear, that is, write in Text in the Maple worksheet what you are doing or finding.

[See Maple](#)

Question 3 (10 points)

John Doe is an avid reader of science fiction, math books and philosophy. He reads one book every week. If he is presently reading science fiction, there is a 60% chance he will switch to a math book next week, and a 40% chance he will switch to philosophy. If he is presently reading a math book, there is a 50% chance that he will read another one the next week, with a 25% percent chance that he will switch to science fiction. If he is presently reading philosophy, there is a 75% chance that he will switch to science fiction the next week, and he never reads two philosophy books in a row.

- a) Find the transition matrix.

$$P = \begin{array}{c} \text{SF} \\ \text{M} \\ \text{P} \end{array} \begin{array}{ccc} \text{SF} & \text{M} & \text{P} \\ \begin{bmatrix} 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{3}{5} & \frac{1}{2} & \frac{1}{4} \\ \frac{2}{5} & \frac{1}{4} & 0 \end{bmatrix} \end{array}$$

- b) If John is currently reading a math book, what is the probability he will also be reading a math book two weeks from now?

$$X_2 = P^2 X_0 = \begin{bmatrix} 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{3}{5} & \frac{1}{2} & \frac{1}{4} \\ \frac{2}{5} & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{3}{5} & \frac{1}{2} & \frac{1}{4} \\ \frac{2}{5} & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{3}{5} & \frac{1}{2} & \frac{1}{4} \\ \frac{2}{5} & \frac{1}{4} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{5}{16} \\ \frac{37}{80} \\ \frac{9}{40} \end{bmatrix}$$

Thus the probability that John will read a math book two weeks from now is $\frac{37}{80} = 46.25\%$.

- c) If he is currently reading a math book, what is the probability he will also be reading a math book
- m
- weeks from now, where
- m
- is the number corresponding to the last 3 digits of your student number? Use Maple, where the answer should be given as a percentage, with three significant digits.

[See Maple](#)

- d) Assuming this trend continues indefinitely, what fraction of the book he reads are math ones?

$$(I - A)X = 0$$

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & -\frac{1}{4} & -\frac{3}{4} & 0 \\ -\frac{3}{5} & \frac{1}{2} & -\frac{1}{4} & 0 \\ -\frac{2}{5} & -\frac{1}{4} & 1 & 0 \end{array} \right] \begin{array}{l} R_2 \rightarrow R_2 + \frac{3}{5}R_1 \\ R_3 \rightarrow R_3 + \frac{2}{5}R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -\frac{1}{4} & -\frac{3}{4} & 0 \\ 0 & \frac{7}{20} & -\frac{7}{10} & 0 \\ 0 & -\frac{7}{20} & \frac{7}{10} & 0 \end{array} \right] \begin{array}{l} R_3 \rightarrow R_3 + R_2 \\ R_2 \rightarrow \frac{20}{7}R_2 \end{array} \left[\begin{array}{ccc|c} 1 & -\frac{1}{4} & -\frac{3}{4} & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$

$$\text{Thus } z = t$$

$$y = 2t$$

$$x = \frac{3}{4}t + \frac{1}{4}2t = \frac{5}{4}t$$

$$\text{Since } x + y + z = 1 \text{ then } \frac{5}{4}t + 2t + t = 1$$

$$\frac{17}{4}t = 1$$

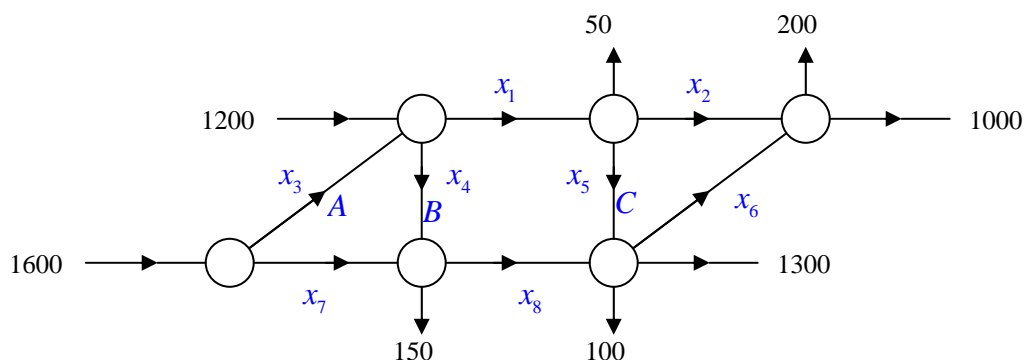
$$t = \frac{4}{17}$$

Thus, $X = \left(\frac{5}{17}, \frac{8}{17}, \frac{4}{17}\right)$, hence, in the long run, $\frac{8}{17} \approx 47.1\%$

- e) Find the eigenvalues of the transition matrix using Maple.
[See Maple](#)
- f) Using Maple, find an eigenvector, whose sum of the entries is one, associated with the biggest eigenvalue found in (d). Compare your answer to what you obtained in (c).
[See Maple](#)

Question 4 (5 points)

Water is flowing through a network of pipes (in cubic meters per hour) as shown below.



- a) Set up the system of linear equations (without evaluating it).

$$\begin{array}{rcll}
 x_1 + x_4 = x_3 + 1200 & \text{Rewriting:} & x_1 & - x_3 + x_4 & = 1200 \\
 x_1 = x_2 + x_5 + 50 & & x_1 - x_2 & - x_5 & = 50 \\
 x_2 + x_6 = 1000 + 200 & & x_2 & + x_6 & = 1200 \\
 x_3 + x_7 = 1600 & & x_3 & + x_7 & = 1600 \\
 x_4 + x_7 = x_8 + 150 & & x_4 & + x_7 - x_8 & = 150 \\
 x_5 + x_8 = x_6 + 1300 + 100 & & x_5 - x_6 & + x_8 & = 1400
 \end{array}$$

- b) Solve the system of linear equations using Maple.

With Maple, we obtain

$$\begin{array}{rcl}
 x_1 = 2650 - t & t \leq 2650 \\
 x_2 = 1200 - r & r \leq 1200 \\
 x_3 = 1600 - s & s \leq 1600 \\
 x_4 = 150 + t - s & s \leq t + 150 \\
 x_5 = 1400 + r - t & t \leq r + 1400 \\
 x_6 = r & r \geq 0 \\
 x_7 = s & s \geq 0 \\
 x_8 = t & t \geq 0
 \end{array}$$

Thus the flow (in m^3/h) is $(2650 - t, 1200 - r, 1600 - s, 150 + t - s, 1400 + r - t, r, s, t)$ where $0 \leq t \leq 2650$, $0 \leq s \leq 1600$ and $0 \leq r \leq 1200$.

- c) What should the flow be if pipes A, B and C are broken and the flow through them is zero?

$$x_3 = 1600 - s = 0 \quad \Rightarrow s = 1600$$

$$x_4 = 150 + t - s = 0 \quad \Rightarrow t = 1450$$

$$x_5 = 1400 + r - t = 0 \quad \Rightarrow r = 50$$

Thus the flow would be:

$$x_1 = 1200 \quad x_4 = 0 \quad x_7 = 1600$$

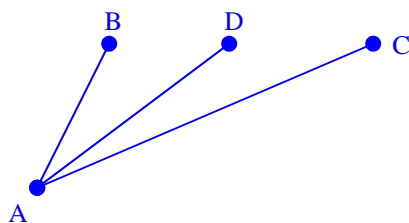
$$x_2 = 1150 \quad x_5 = 0 \quad x_8 = 1450$$

$$x_3 = 0 \quad x_6 = 50$$

Question 5 (5 points)

Let A , B and C be three points, not in a straight line such that $\overrightarrow{AD} = \frac{2}{7}\overrightarrow{AB} + \frac{5}{7}\overrightarrow{AC}$.

- a) Prove that \overrightarrow{BC} and \overrightarrow{CD} are parallel vectors.



$$\begin{aligned} \overrightarrow{CD} &= \overrightarrow{CA} + \overrightarrow{AD} \\ &= -\overrightarrow{AC} + \frac{2}{7}\overrightarrow{AB} + \frac{5}{7}\overrightarrow{AC} \\ &= \frac{2}{7}\overrightarrow{AB} - \frac{2}{7}\overrightarrow{AC} \\ &= \frac{2}{7}\overrightarrow{AB} - \frac{2}{7}(\overrightarrow{AB} + \overrightarrow{BC}) \\ &= -\frac{2}{7}\overrightarrow{BC} \end{aligned}$$

Since \overrightarrow{CD} is a scalar multiple of \overrightarrow{BC} , then \overrightarrow{BC} and \overrightarrow{CD} are parallel vectors.

- b) If $\|\overrightarrow{BC}\| = 21$, find $\|\overrightarrow{CD}\|$.

$$\begin{aligned} \|\overrightarrow{CD}\| &= \left\| -\frac{2}{7}\overrightarrow{BC} \right\| \\ &= \frac{2}{7}\|\overrightarrow{BC}\| \\ &= \frac{2}{7}21 = 6 \end{aligned}$$