MATHEMATICS 201-NYC-05
Vectors and Matrices
Martin Huard
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Assignment #1
SOLUTIONS

This assignment is due ***Friday September 7*** at the beginning of the class.
For questions involving Maple (questions 5 and 6), a print-out of your work is expected, where your name is written in the Worksheet, and each question is clearly labeled.

Question 1 (9 points)

Let P be an $n\times 1$ matrix such that $P^TP=1$. We define the ***Householder matrix*** H as $H=I-2PP^T$.

a) Let $P=\begin{bmatrix}\frac{2}{3}\\ \frac{1}{3}\\ \frac{-2}{3}\end{bmatrix}$. Verify that $P^TP=1$ and find H.

$$P^TP=\begin{bmatrix}\frac{2}{3} & \frac{1}{3} & \frac{-2}{3}\end{bmatrix}\begin{bmatrix}\frac{2}{3}\\ \frac{1}{3}\\ \frac{-2}{3}\end{bmatrix}=\begin{bmatrix}\frac{4}{9}+\frac{1}{9}+\frac{4}{9}\end{bmatrix}=\begin{bmatrix}1\end{bmatrix}=I$$

$$H=I-2PP^T=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-2\begin{bmatrix}\frac{2}{3}\\ \frac{1}{3}\\ \frac{-2}{3}\end{bmatrix}\begin{bmatrix}\frac{2}{3} & \frac{1}{3} & \frac{-2}{3}\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-2\begin{bmatrix}\frac{4}{9}&\frac{2}{9}&\frac{-4}{9}\\ \frac{2}{9}&\frac{1}{9}&\frac{-2}{9}\\ \frac{-4}{9}&\frac{-2}{9}&\frac{4}{9}\end{bmatrix}=\begin{bmatrix}\frac{1}{9}&\frac{-4}{9}&\frac{8}{9}\\ \frac{-4}{9}&\frac{7}{9}&\frac{4}{9}\\ \frac{8}{9}&\frac{4}{9}&\frac{1}{9}\end{bmatrix}$$

b) Prove that if H is any Householder matrix, then H is symmetric.

To prove: $H^T=H$

$$\begin{aligned}LS&=H^T\\&=\left(I-2PP^T\right)^T\\&=I^T-2\left(P^T\right)^TP^T\\&=I-2PP^T\\&=H=RS\end{aligned}$$

c) Prove that if H is any Householder matrix, then $H^{-1} = H^T$.

To prove: $H^T H = I$

$$
\begin{aligned}
LS &= H^T H \\
&= HH \qquad \text{since } H \text{ is symmetric} \\
&= \left(I - 2PP^T\right)\left(I - 2PP^T\right) \\
&= II - 2PP^T I - 2IPP^T + 4PP^T PP^T \\
&= I - 2PP^T - 2PP^T + 4P\left(P^T P\right)P^T \\
&= I - 4PP^T + 4PP^T \qquad \text{since } P^T P = I \\
&= I = RS
\end{aligned}
$$

Question 2 (10 points)

Solve for *x*, *y* and *z*.

$$
\begin{aligned}
xy - 2\sqrt{y} + 3zy &= 8 \\
2xy - 3\sqrt{y} + 2zy &= 7 \\
-xy + \sqrt{y} + 2zy &= 5
\end{aligned}
$$

Let $u = xy$, $v = \sqrt{y}$ and $w = zy$. Then the above system becomes

$$
\begin{aligned}
u - 2v + 3w &= 8 \\
2u - 3v + 2w &= 7 \\
-u + v + 2w &= 5
\end{aligned}
$$

$$
\left[\begin{array}{ccc|c} 1 & -2 & 3 & 8 \\ 2 & -3 & 2 & 7 \\ -1 & 1 & 2 & 5 \end{array}\right]
\xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1}
\left[\begin{array}{ccc|c} 1 & -2 & 3 & 8 \\ 0 & 1 & -4 & -9 \\ 0 & -1 & 5 & 13 \end{array}\right]
$$

$$
\xrightarrow{R_3 \to R_3 + R_2}
\left[\begin{array}{ccc|c} 1 & -2 & 3 & 8 \\ 0 & 1 & -4 & -9 \\ 0 & 0 & 1 & 4 \end{array}\right]
$$

Hence $w = 4$, $v = 7$ and $u = 10$.

Thus $y = 49, x = \frac{10}{49}$ and $z = \frac{4}{49}$, and the solution is $\left(\frac{10}{49}, 49, \frac{4}{49}\right)$.

Question 3 (10 points)

For which values of a will the following system of linear equations have

$$\begin{aligned} a^2x - 6ay + 12z &= 8 \\ 2x - 3y + 5z &= 2 \\ x - 3y + 3z &= 2 \end{aligned}$$

i) a unique solution
ii) no solution
iii) an infinite number of solutions

$$\left[\begin{array}{ccc|c} a^2 & -6a & 12 & 8 \\ 2 & -3 & 5 & 2 \\ 1 & -3 & 3 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -3 & 3 & 2 \\ 2 & -3 & 5 & 2 \\ a^2 & -6a & 12 & 8 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - a^2R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -3 & 3 & 2 \\ 0 & 3 & -1 & -2 \\ 0 & -6a+3a^2 & 12-3a^2 & 8-2a^2 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 + \left(2a - a^2\right)R_2} \left[\begin{array}{ccc|c} 1 & -3 & 3 & 2 \\ 0 & 3 & -1 & -2 \\ 0 & 0 & -2a^2-2a+12 & 8-4a \end{array}\right]$$

$$\xrightarrow[R_3 \to \frac{1}{-2a^2-2a+12}R_3]{R_2 \to \frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & -3 & 3 & 2 \\ 0 & 1 & \frac{-1}{3} & \frac{-2}{3} \\ 0 & 0 & 1 & \frac{8-4a}{-2a^2-2a+12} \end{array}\right]$$

Assuming $-2a^2 - 2a + 12 \neq 0$

$$-2\left(a^2 + a - 6\right) \neq 0$$

$$a \neq -3, 2$$

Hence, if $a \neq -3$ and $a \neq 2$ then the system has a unique solution.

If $a = -3$, then we have

$$\left[\begin{array}{ccc|c} 1 & -3 & 3 & 2 \\ 0 & 3 & -1 & -2 \\ 0 & 0 & 0 & 20 \end{array}\right]$$

so our system has no solutions.

If $a = 2$, then we have

$$\left[\begin{array}{ccc|c} 1 & -3 & 3 & 2 \\ 0 & 3 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} z &= t \\ y &= -2 + t \\ x &= -8 + 3t \end{aligned}$$

so our system has an infinite number of solutions

Question 4 (10 points)

Find A such that $\left(I+2A^T\right)^{-1}=\begin{bmatrix}2 & 3\\ -1 & -1\end{bmatrix}^2$.

$$\begin{bmatrix}2 & 3\\ -1 & -1\end{bmatrix}^2=\begin{bmatrix}2 & 3\\ -1 & -1\end{bmatrix}\begin{bmatrix}2 & 3\\ -1 & -1\end{bmatrix}=\begin{bmatrix}1 & 3\\ -1 & -2\end{bmatrix}$$

Thus $I+2A^T=\begin{bmatrix}1 & 3\\ -1 & -2\end{bmatrix}^{-1}$. Finding the inverse,

$$\left[\begin{array}{cc|cc}1 & 3 & 1 & 0\\ -1 & -2 & 0 & 1\end{array}\right]\xrightarrow{R_2\to R_2+R_1}\left[\begin{array}{cc|cc}1 & 3 & 1 & 0\\ 0 & 1 & 1 & 1\end{array}\right]\xrightarrow{R_1\to R_1-3R_1}\left[\begin{array}{cc|cc}1 & 3 & -2 & -3\\ 0 & 1 & 1 & 1\end{array}\right]$$

Ergo, $I+2A^T=\begin{bmatrix}-2 & -3\\ 1 & 1\end{bmatrix}$

$$2A^T=\begin{bmatrix}-2 & -3\\ 1 & 1\end{bmatrix}-\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}-3 & -3\\ 1 & 0\end{bmatrix}$$

$$A^T=\tfrac{1}{2}\begin{bmatrix}-3 & -3\\ 1 & 0\end{bmatrix}=\begin{bmatrix}\frac{-3}{2} & \frac{-3}{2}\\ \frac{1}{2} & 0\end{bmatrix}$$

$$A=\begin{bmatrix}\frac{-3}{2} & \frac{-3}{2}\\ \frac{1}{2} & 0\end{bmatrix}^T=\begin{bmatrix}\frac{-3}{2} & \frac{1}{2}\\ \frac{-3}{2} & 0\end{bmatrix}$$

Question 5 (7 points)

Let $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \\ -2 & 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 11 & -1 & -1 \\ 3 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 2 \\ 3 & -2 \\ 1 & 4 \end{bmatrix}$.

Using Maple, evaluate the following.

a) $6A^{-1}B$

b) $\left(A^2B^{-1}C\right)^T$

c) $(A+B)^2$

d) $A^2 + 2AB + B^2$

e) Compare your answers in (c) and (d). Why is $(A+B)^2 \neq A^2 + 2AB + B^2$ for matrices? Justify your answer! (Type your answer on the Maple worksheet!)

Question 6 (4 points)

Using Taylor Series, we can define various analytical functions. For example, in Calculus 2, we saw that the Taylor Series for e^x is given by

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

This leads to the following definition for the matrix exponential

$$e^A = I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \cdots$$

Using this definition,

a) With Maple, approximate the matrix e^I using the first 20 terms of the Taylor Series, where I is the 2×2 identity matrix. Give the answer in decimal form.

b) Let $B = e^I$. Guess what the matrix B should be, if an infinite sum is used, based on your answer in (a). (Type your answer on the Maple worksheet!)