



MATHEMATICS 201-NYB-05

Integral Calculus

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VI - Integration by Parts

1. Evaluate the integral.

a) $\int x e^{-x} dx$

b) $\int x^2 e^x dx$

c) $\int x \cos 3x dx$

d) $\int x^2 \sin x dx$

e) $\int \sqrt{x} \ln x dx$

f) $\int (\ln x)^2 dx$

g) $\int \frac{x e^x}{(1+x)^2} dx$

h) $\int e^{2x} \cos 3x dx$

i) $\int e^{ax} \sin bx dx$

j) $\int \cos(\ln x) dx$

k) $\int \frac{\ln x}{x^2} dx$

l) $\int \sin 3\theta \cos 5\theta d\theta$

m) $\int \arcsin 2x dx$

n) $\int e^{\sqrt{t}} dt$

o) $\int x \tan^2 x dx$

2. Evaluate the given integral.

a) $\int_0^1 x e^{2x} dx$

b) $\int_1^{e^2} x \ln \sqrt{x} dx$

c) $\int_0^{\frac{\pi}{2}} 4x^2 \sin 3x dx$

d) $\int_0^2 \frac{x+2}{e^{3x}} dx$

e) $\int_0^{\frac{\pi}{2}} x \sin^2 x dx$

f) $\int_{-1}^1 \arccos x dx$

g) $\int_{11}^{18} r \sqrt{r-2} dr$

h) $\int_0^1 x \arctan(x^2) dx$

i) $\int_2^4 \operatorname{arcsec} \sqrt{x} dx$

3. Prove the following reduction formulas.

a) $\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$

b) $\int (\ln x)^n dx = x (\ln x)^n - n \int (\ln x)^{n-1} dx$

c) $\int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx$

d) $\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx$

4. Use the reduction formulas proved in Question 3 to evaluate the following integrals.

a) $\int x^6 e^x dx$

b) $\int (\ln x)^4 dx$

c) $\int \cos^5 x dx$

d) $\int_0^{\frac{\pi}{4}} \tan^6 x dx$

5. Use a reduction formula from Question 3 to prove $\int_0^{\frac{\pi}{2}} \cos^{2n} x dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \frac{\pi}{2}$.

6. Prove that if $f(0) = g(0) = 0$ and f'' and g'' are continuous, then

$$\int_0^t f(x) g''(x) dx = f(t) g'(t) - f'(t) g(t) + \int_0^t f''(x) g(x) dx$$

Answers

1. a) $-xe^{-x} - e^{-x} + C$ b) $x^2e^x - 2xe^x + 2e^x + C$
 c) $\frac{1}{3}x\sin(3x) + \frac{1}{9}\cos(3x) + C$ d) $-x^2\cos x + 2x\sin x + 2\cos x + C$
 e) $\frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4}{9}x^{\frac{3}{2}} + C$ f) $x(\ln x)^2 - 2x\ln x + 2x + C$
 g) $\frac{e^x}{1+x} + C$ h) $\frac{2}{13}e^{2x}\cos 3x + \frac{3}{13}e^{2x}\sin 3x + C$
 i) $\frac{a}{a^2+b^2}e^{ax}\sin bx - \frac{b}{a^2+b^2}e^{ax}\cos bx + C$ j) $\frac{1}{2}x\cos(\ln x) + \frac{1}{2}x\sin(\ln x) + C$
 k) $-\frac{\ln x}{x} - \frac{1}{x} + C$ l) $\frac{5}{16}\sin 3\theta\sin 5\theta + \frac{3}{16}\cos 3\theta\cos 5\theta + C$
 m) $x\arcsin(2x) + \frac{1}{2}\sqrt{1-4x^2} + C$ n) $2\sqrt{t}e^{\sqrt{t}} - 2e^{\sqrt{t}} + C$
 o) $x\tan x - \frac{1}{2}x^2 + \ln|\cos x| + C$
2. a) $\frac{1}{4}e^2 + \frac{1}{4}$ b) $\frac{3}{8}e^4 + \frac{1}{8}$ c) $-\frac{2}{243}\pi^2 - \frac{4}{27} + \frac{4}{81}\pi\sqrt{3}$
 d) $\frac{7}{9} - \frac{13}{9e^6}$ e) $\frac{1}{16}\pi^2 + \frac{1}{4}$ f) π
 g) $\frac{5426}{15}$ h) $\frac{\pi}{8} - \frac{1}{4}\ln 2$ i) $\frac{5\pi}{6} - \sqrt{3} + 1$
4. a) $(x^6 - 6x^5 + 30x^4 - 120x^3 + 360x^2 - 720x + 720)e^x + C$
 b) $x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24x\ln x + 24x + C$
 c) $\frac{1}{5}\cos^4 x \sin x + \frac{4}{15}\cos^2 x \sin x + \frac{8}{15}\sin x + C$
 d) $\frac{13}{15} - \frac{\pi}{4}$
5. $\int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx = \left[\frac{1}{2n} \cos^{2n-1} x \sin x \right]_0^{\frac{\pi}{2}} + \frac{2n-1}{2n} \int_0^{\frac{\pi}{2}} \cos^{2n-2} x \, dx$ applying reduction formula
 $= (0-0) + \frac{2n-1}{2n} \left[\frac{1}{2n-2} \cos^{2n-3} x \sin x \right]_0^{\frac{\pi}{2}} + \frac{2n-3}{2n-2} \int_0^{\frac{\pi}{2}} \cos^{2n-4} x \, dx$
 $= \frac{(2n-1)(2n-3)}{2n(2n-2)} \int_0^{\frac{\pi}{2}} \cos^{2n-4} x \, dx$
 ...
 $= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \int_0^{\frac{\pi}{2}} dx$
 $= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \frac{\pi}{2}$
6. $\int_0^t f(x)g''(x)dx = [f(x)g'(x)]_0^t - \int_0^t f'(x)g'(x)dx$ $u = f(x)$ $v = g'(x)$
 $= f(t)g'(t) - [f'(x)g(x)]_0^t + \int_0^t f''(x)g(x)dx$ $\frac{du = f'(x)dx}{u = f'(x)}$ $\frac{dv = g''(x)dx}{v = g(x)}$
 $= f(t)g'(t) - f'(t)g(t) + \int_0^t f''(x)g(x)dx$ $\frac{du = f''(x)dx}{dv = g'(x)dx}$