



MATHEMATICS 201-NYA-05

Differential Calculus

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IV - Continuity

SOLUTIONS

1. Examine the continuity of f at $x=a$. If f is discontinuous at $x=a$, state the kind of discontinuity.

$$\text{a) } f(x) = \begin{cases} \frac{x^2 - x - 6}{x - 3} & x \neq 3 \\ 5 & x = 3 \end{cases} \quad \text{at } x = 3$$

$$1. f(3) = 5$$

$$\begin{aligned} 2. \lim_{x \rightarrow 3} f(x) &= \lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{\cancel{(x-3)}(x+2)}{\cancel{x-3}} && x \neq 3 \\ &= \lim_{x \rightarrow 3} (x+2) \\ &= 5 \end{aligned}$$

$$3. \lim_{x \rightarrow 3} f(x) = 5 = f(3)$$

Thus f is continuous at $x = 3$.

$$\text{b) } f(x) = \begin{cases} \frac{x^2 + x - 6}{x - 3} & x \neq 3 \\ 5 & x = 3 \end{cases} \quad \text{at } x = 3$$

$$1. f(3) = 5$$

$$\begin{aligned} 2. \lim_{x \rightarrow 3^-} f(x) &= \lim_{x \rightarrow 3^-} \frac{x^2 + x - 6}{x - 3} && x < 3 \\ &= -\infty && \text{case } \frac{b}{0} \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 3^+} f(x) &= \lim_{x \rightarrow 3^+} \frac{x^2 + x - 6}{x - 3} && x > 3 \\ &= \infty && \text{case } \frac{b}{0} \end{aligned}$$

$$\text{Thus } \lim_{x \rightarrow 3} f(x) \nexists$$

Hence f has an infinite discontinuity at $x = 3$.

$$c) f(x) = \begin{cases} \frac{x^2 - x - 6}{x - 3} & x \neq 3 \\ 3 & x = 3 \end{cases} \quad \text{at } x = 3$$

$$1. f(3) = 3$$

$$2. \lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x - 3}$$

$$= \lim_{x \rightarrow 3} \frac{\cancel{(x-3)}(x+2)}{\cancel{x-3}} \quad x \neq 3$$

$$= \lim_{x \rightarrow 3} (x+2)$$

$$= 5$$

$$3. \lim_{x \rightarrow 3} f(x) = 5 \neq f(3) = 3$$

Thus f has a removable discontinuity at $x = 3$.

$$d) f(x) = \begin{cases} x^2 + 1 & x \leq 2 \\ 3x - 4 & x > 2 \end{cases} \quad \text{at } x = 2$$

$$1. f(2) = 2^2 + 1 = 5$$

$$2. \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x^2 + 1) \quad x < 2 \quad \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (3x - 4) \quad x > 2$$

$$= 5 \quad = 11$$

$$\therefore \lim_{x \rightarrow 2} f(x) \nexists$$

Thus f has a jump discontinuity at $x = 2$.

$$e) f(x) = \begin{cases} 4 + x & x < 1 \\ x^2 + 4 & x \geq 1 \end{cases} \quad \text{at } x = 1$$

$$1. f(1) = 1^2 + 4 = 5$$

$$2. \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (4 + x) \quad x < 1 \quad \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x^2 + 4) \quad x > 1$$

$$= 5 \quad = 5$$

$$\therefore \lim_{x \rightarrow 1} f(x) = 5$$

$$3. \lim_{x \rightarrow 1} f(x) = 5 = f(1)$$

Thus f is continuous at $x = 1$.

$$\text{f) } f(x) = \begin{cases} |x-3| & x < 3 \\ x-3 & \\ \frac{1}{x-4} & x \geq 3 \end{cases} \quad \text{at } x=3$$

$$1. f(3) = \frac{1}{3-4} = -1$$

$$\begin{aligned} 2. \lim_{x \rightarrow 3^-} f(x) &= \lim_{x \rightarrow 3^-} \frac{|x-3|}{x-3} \quad x < 3 \\ &= \lim_{x \rightarrow 3^-} \frac{\cancel{-(x-3)}}{\cancel{x-3}} \\ &= \lim_{x \rightarrow 3^-} -1 \\ &= -1 \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 3^+} f(x) &= \lim_{x \rightarrow 3^+} \frac{1}{x-4} \quad x > 3 \\ &= -1 \end{aligned}$$

$$\therefore \lim_{x \rightarrow 3} f(x) = -1$$

$$3. \lim_{x \rightarrow 3} f(x) = -1 = f(3)$$

Thus f is continuous at $x=3$.

$$\text{g) } f(x) = \begin{cases} 3+x^2 & x < 2 \\ 0 & x = 2 \\ 11-x^2 & x > 2 \end{cases} \quad \text{at } x=2$$

$$1. f(2) = 0$$

$$\begin{aligned} 2. \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} (3+x^2) \quad x < 2 \\ &= 7 \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 2^+} f(x) &= \lim_{x \rightarrow 2^+} (11-x^2) \quad x > 2 \\ &= 7 \end{aligned}$$

$$\therefore \lim_{x \rightarrow 2} f(x) = 7$$

$$3. \lim_{x \rightarrow 2} f(x) = 7 \neq f(2) = 0$$

Thus f has a removable discontinuity at $x=2$.

$$\text{h) } f(x) = \begin{cases} \llbracket x \rrbracket & x \leq 4 \\ \frac{x^2 - 5x + 4}{x - 4} & x > 4 \end{cases} \quad \text{at } x = 4$$

$$1. f(4) = \llbracket 4 \rrbracket = 4$$

$$2. \lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^-} \llbracket 4 \rrbracket \quad x < 4 \\ = 3$$

$$\begin{aligned} \lim_{x \rightarrow 4^+} f(x) &= \lim_{x \rightarrow 4^+} \frac{x^2 - 5x + 4}{x - 4} \quad x > 4 \\ &= \lim_{x \rightarrow 4^+} \frac{(x-4)(x-1)}{x-4} \\ &= \lim_{x \rightarrow 4^+} (x-1) \\ &= 3 \end{aligned}$$

$$\therefore \lim_{x \rightarrow 4} f(x) = 3$$

$$3. \lim_{x \rightarrow 4} f(x) = 3 \neq f(4) = 4$$

Thus f has a removable discontinuity at $x = 4$.

$$\text{i) } f(x) = \llbracket 1-x \rrbracket + \llbracket x-1 \rrbracket \quad \text{at } x = 1$$

$$1. f(1) = \llbracket 1-1 \rrbracket + \llbracket 1-1 \rrbracket = 0 + 0 = 0$$

$$2. \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (\llbracket 1-x \rrbracket + \llbracket x-1 \rrbracket) \quad x < 1 \\ = 0 + (-1) = -1$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (\llbracket 1-x \rrbracket + \llbracket x-1 \rrbracket) \quad x > 1 \\ = -1 + 0 = -1$$

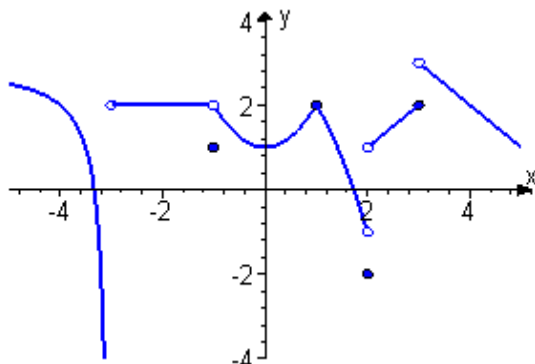
$$\therefore \lim_{x \rightarrow 1} f(x) = -1$$

$$3. \lim_{x \rightarrow 1} f(x) = -1 \neq f(1) = 0$$

Thus f has a removable discontinuity at $x = 1$.

2. For each of the functions f whose graph is given below, find the points where f is discontinuous and state the type of discontinuity and find the intervals on which f is continuous.

a)



Since $\lim_{x \rightarrow -3^-} f(x) = -\infty$, then f has an infinite discontinuity at $x = -3$.

Since $\lim_{x \rightarrow -1} f(x) = 2 \neq f(-1) = 1$, then f has a removable discontinuity at $x = -1$

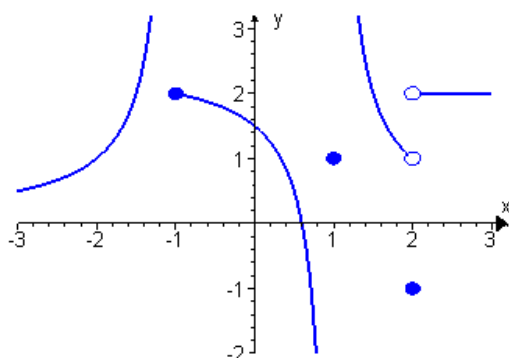
Since $\lim_{x \rightarrow 2^-} f(x) = -1 \neq \lim_{x \rightarrow 2^+} f(x) = 1$, then f has a jump discontinuity at $x = 2$

Since $\lim_{x \rightarrow 3^-} f(x) = 2 \neq \lim_{x \rightarrow 3^+} f(x) = 3$, then f has a jump discontinuity at $x = 3$

Since $\lim_{x \rightarrow 3^-} f(x) = 2 = f(3)$, then f is continuous from the left at $x = 3$

Ergo, f is continuous on $(-\infty, -3), (-3, -1), (-1, 2), (2, 3], (3, \infty)$

b)



Since $\lim_{x \rightarrow -1^-} f(x) = -\infty$, then f has an infinite discontinuity at $x = -1$.

Since $\lim_{x \rightarrow 1^+} f(x) = \infty$, then f has an infinite discontinuity at $x = 1$

Since $\lim_{x \rightarrow 2^-} f(x) = 1 \neq \lim_{x \rightarrow 2^+} f(x) = 2$, then f has a jump discontinuity at $x = 2$

Since $\lim_{x \rightarrow -1^+} f(x) = 2 = f(-1)$, then f is continuous from the right at $x = -1$

Ergo, f is continuous on $(-\infty, -1), [-1, 1), (1, 2), (2, \infty)$

3. Discuss the continuity of the following functions. If the function is discontinuous at a point, state the kind of discontinuity.

a) $f(x) = \frac{x^2 + x - 6}{x^2 - 4}$

The function $f(x)$ is a rational function, thus is continuous on its domain

$$\mathbb{R} / \{-2, 2\}.$$

At $x = -2$,

$$\lim_{x \rightarrow -2^-} \frac{x^2 + x - 6}{x^2 - 4} = -\infty \quad \text{case } \frac{b}{0}$$

$$\lim_{x \rightarrow -2^+} \frac{x^2 + x - 6}{x^2 - 4} = \infty \quad \text{case } \frac{b}{0}$$

So f has an infinite discontinuity at $x = -2$.

At $x = 2$,

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^2 + x - 6}{x^2 - 4} &= \lim_{x \rightarrow 2} \frac{(x+3)(x-2)}{(x-2)(x+2)} \\ &= \lim_{x \rightarrow 2} \frac{x+3}{x+2} \quad x \neq 2 \\ &= \frac{5}{4} \end{aligned}$$

So f has a removable discontinuity $x = 2$.

Hence f is continuous on $(-\infty, -2)$, $(-2, 2)$ and $(2, \infty)$ with a removable discontinuity at $x = 2$ and an infinity discontinuity at $x = -2$.

b) $f(x) = \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x^2-1}$

The function $f(x)$ is the sum of three rational functions, $\frac{1}{x}$ which is continuous on its domain $\mathbb{R} / \{0\}$, $\frac{1}{x-1}$ which is continuous on its domain $\mathbb{R} / \{1\}$ and $\frac{1}{x^2-1}$ which is continuous on its domain $\mathbb{R} / \{-1, 1\}$. Thus $f(x)$ is continuous on $\mathbb{R} / \{-1, 0, 1\}$.

At $x = -1$,

$$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x^2-1} \right) = -\frac{3}{2} + \lim_{x \rightarrow -1^-} \frac{1}{x^2-1} = \infty \quad \text{case } \frac{b}{0}$$

$$\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x^2-1} \right) = -\frac{3}{2} + \lim_{x \rightarrow -1^+} \frac{1}{x^2-1} = -\infty \quad \text{case } \frac{b}{0}$$

So f has an infinite discontinuity at $x = -1$.

At $x = 0$,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x^2-1} \right) = \lim_{x \rightarrow 0^-} \frac{1}{x} - 2 = -\infty \quad \text{case } \frac{b}{0}$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x^2-1} \right) = \lim_{x \rightarrow 0^+} \frac{1}{x} - 2 = \infty \quad \text{case } \frac{b}{0}$$

So f has an infinite discontinuity at $x = 0$.

At $x = 1$,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x^2-1} \right) = 1 + \lim_{x \rightarrow 1^-} \frac{x+1+1}{x^2-1} = -\infty \quad \text{case } \frac{b}{0}$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x^2-1} \right) = 1 + \lim_{x \rightarrow 1^+} \frac{x+1+1}{x^2-1} = \infty \quad \text{case } \frac{b}{0}$$

So f has an infinite discontinuity at $x = 1$.

Ergo, f is continuous on $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$ and $(1, \infty)$ with infinity discontinuities at $x = 0, \pm 1$

a) $f(x) = \sqrt{\frac{1}{x^2+1}}$

The function $f(x)$ is continuous on \mathbb{R} since it is the composition of two continuous functions, $\frac{1}{x^2+1}$, a rational function continuous on its domain \mathbb{R} , and \sqrt{x} the square root function which is also continuous since $\frac{1}{x^2+1} > 0$ for all values of x .

b) $f(x) = |x^2 + 2x + 1|$

The function $f(x)$ is continuous on \mathbb{R} since it is the composition of two continuous function on \mathbb{R} , $x^2 + 2x + 1$, a polynomial continuous on its domain \mathbb{R} , and $|x|$ the absolute value function also continuous on its domain \mathbb{R} .

e) $f(x) = \begin{cases} \frac{x^2 - x - 6}{x - 3} & x \neq 3 \\ 3 & x = 3 \end{cases}$

On $\mathbb{R}/\{3\}$, $f(x) = \frac{x^2 - x - 6}{x - 3}$ which is a rational function, thus continuous on its domain $\mathbb{R}/\{3\}$.

At $x = 3$

1. $f(3) = 3$

$$\begin{aligned}
 2. \quad \lim_{x \rightarrow 3} f(x) &= \lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x - 3} \\
 &= \lim_{x \rightarrow 3} \frac{\cancel{(x-3)}(x+2)}{\cancel{x-3}} \quad x \neq 3 \\
 &= \lim_{x \rightarrow 3} (x+2) \\
 &= 5 \\
 3. \quad \lim_{x \rightarrow 3} f(x) &= 5 \neq f(3) = 3
 \end{aligned}$$

Thus f has a removable discontinuity at $x = 3$.

Ergo, f is continuous on $(-\infty, 3)$ and $(3, \infty)$ with a removable discontinuity at $x = 3$

$$f) \quad f(x) = \begin{cases} \frac{x}{x^2 + 1} & x \leq 1 \\ \frac{1}{x - 5} & x > 1 \end{cases}$$

On $(-\infty, 1)$, $f(x) = \frac{x}{x^2 + 1}$ which is a rational function, hence continuous on its domain \mathbb{R} , thus continuous on $(-\infty, 1)$.

On $(1, \infty)$, $f(x) = \frac{1}{x - 5}$ which is a rational function, hence continuous on its domain $\mathbb{R} / \{5\}$, thus continuous on $(1, 5)$ and $(5, \infty)$.

At $x = 5$

$$\begin{aligned}
 \lim_{x \rightarrow 5^-} f(x) &= \lim_{x \rightarrow 5^-} \frac{1}{x - 5} = -\infty \quad \text{case } \frac{b}{0} \\
 \lim_{x \rightarrow 5^+} f(x) &= \lim_{x \rightarrow 5^+} \frac{1}{x - 5} = \infty \quad \text{case } \frac{b}{0} \\
 \therefore \lim_{x \rightarrow 5} f(x) &\not\exists
 \end{aligned}$$

Thus f has an infinite discontinuity at $x = 5$

At $x = 1$

$$\begin{aligned}
 1. \quad f(1) &= \frac{1}{1^2 + 1} = \frac{1}{2} \\
 2. \quad \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} \frac{x}{x^2 + 1} \quad x < 1 & \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} \frac{1}{x - 5} \quad x > 1 \\
 &= \frac{1}{2} & &= \frac{-1}{4} \\
 \therefore \lim_{x \rightarrow 1} f(x) &\not\exists
 \end{aligned}$$

Thus f has a jump discontinuity at $x = 1$, but since $\lim_{x \rightarrow 1^-} f(x) = \frac{1}{2} = f(1)$, then f is continuous from the left at $x = 1$.

Hence f is continuous on $(-\infty, 1]$, $(1, 5)$ and $(5, \infty)$ with a jump discontinuity at $x = 1$ and an infinite discontinuity at $x = 5$.

$$\text{g) } f(x) = \begin{cases} x^2 - 1 & x < 3 \\ 2x + 2 & x \geq 3 \end{cases}$$

On $(-\infty, 3)$, $f(x) = x^2 - 1$ which is a polynomial function, hence continuous on \mathbb{R} , thus continuous on $(-\infty, 3)$.

On $(3, \infty)$, $f(x) = 2x + 2$ which is a polynomial function, hence continuous on \mathbb{R} , thus continuous on $(3, \infty)$.

At $x = 3$

$$1. \quad f(3) = 2 \cdot 3 + 2 = 8$$

$$2. \quad \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} (x^2 - 1) \quad x < 3 \qquad \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} (2x + 2) \quad x > 3$$

$$= 8 \qquad \qquad \qquad = 8$$

$$\therefore \lim_{x \rightarrow 3} f(x) = 8$$

$$3. \quad \therefore \lim_{x \rightarrow 3} f(x) = 8 = f(3)$$

Thus f is continuous at $x = 3$.

Hence f is continuous on \mathbb{R} .

$$\text{h) } f(x) = \begin{cases} 3x + 1 & x < -2 \\ \frac{x^2 - 1}{x} & x \geq -2 \end{cases}$$

On $(-\infty, -2)$, $f(x) = 3x + 1$ which is a polynomial function, hence continuous on \mathbb{R} , thus continuous on $(-\infty, -2)$.

On $(-2, \infty)$, $f(x) = \frac{x^2 - 1}{x}$ which is a rational function, hence continuous on its domain $\mathbb{R} / \{0\}$, thus continuous on $(-2, 0), (0, \infty)$.

At $x = 0$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{x^2 - 1}{x} = \infty \qquad \text{case } \frac{b}{0}$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{x^2 - 1}{x} = -\infty \qquad \text{case } \frac{b}{0}$$

$$\therefore \lim_{x \rightarrow 0} f(x) \nexists$$

Thus f has an infinite discontinuity at $x = 0$

At $x = -2$

$$1. \quad f(-2) = \frac{(-2)^2 - 1}{-2} = \frac{-3}{2}$$

$$2. \quad \lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} (3x+1) \quad x < -2 \qquad \lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^+} \frac{x^2-1}{x} \quad x > -2$$

$$= -5 \qquad \qquad \qquad = \frac{-3}{2}$$

$$\therefore \lim_{x \rightarrow -2} f(x) \nexists$$

Hence f has a jump discontinuity at $x = -2$

Since $\lim_{x \rightarrow -2^+} f(x) = \frac{-3}{2} = f(-2)$, then f is continuous from the right at $x = -2$

Ergo, f is continuous on $(-\infty, -2), [-2, 0), (0, \infty)$ with a jump discontinuity at $x = -2$ and an infinite discontinuity at $x = 0$

$$i) \quad f(x) = \begin{cases} \frac{x^2+4}{x^2+2} & x \leq 0 \\ \frac{x^2-4}{x^2-2x} & x > 0 \end{cases}$$

On $(-\infty, 0)$, $f(x) = \frac{x^2+4}{x^2+2}$ which is a rational function, hence continuous on its domain \mathbb{R} , thus continuous on $(-\infty, 0)$.

On $(0, \infty)$, $f(x) = \frac{x^2-4}{x^2-2x}$ which is a rational function, hence continuous on its domain $\mathbb{R} / \{0, 2\}$, thus continuous on $(0, 2), (2, \infty)$.

At $x = 2$

$$\begin{aligned} \lim_{x \rightarrow 2} f(x) &= \lim_{x \rightarrow 2} \frac{x^2-4}{x^2-2x} \\ &= \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{x(x-2)} \\ &= \lim_{x \rightarrow 2} \frac{x+2}{x} \quad x \neq 2 \\ &= 2 \end{aligned}$$

Thus f has a removable discontinuity at $x = 2$

At $x = 0$

$$1. \quad f(0) = \frac{0+4}{0+2} = 2$$

$$2. \quad \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow -2^-} \frac{x^2+4}{x^2+2} \quad x < 0 \qquad \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{x^2-4}{x^2-2x} \quad x > 0$$

$$= 2 \qquad \qquad \qquad = \infty \qquad \text{case } \frac{b}{0}$$

$$\therefore \lim_{x \rightarrow 0} f(x) \nexists$$

Hence f has an infinite discontinuity at $x = 0$

Since $\lim_{x \rightarrow 0^-} f(x) = 2 = f(0)$, then f is continuous from the left at $x = 0$

Ergo, f is continuous on $(-\infty, 0], (0, 2), (2, \infty)$ with an infinite discontinuity at $x = 0$ and a removable discontinuity at $x = 2$.

$$j) \quad f(x) = \begin{cases} \frac{1}{x+2} & x \leq 1 \\ \frac{\sqrt{x}-2}{x-4} & x > 1 \end{cases}$$

On $(-\infty, 1)$, $f(x) = \frac{x^2+4}{x^2+2}$ which is a rational function, hence continuous on its domain $\mathbb{R} / \{-2\}$, thus continuous on $(-\infty, -2), (-2, 1)$.

At $x = -2$

$$\lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} \frac{1}{x+2} = -\infty \quad \text{case } \frac{b}{0}$$

$$\lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^+} \frac{1}{x+2} = \infty \quad \text{case } \frac{b}{0}$$

$$\therefore \lim_{x \rightarrow -2} f(x) \nexists$$

Thus f has an infinite discontinuity at $x = -2$

On $(1, \infty)$, $f(x) = \frac{x^2-4}{x^2-2x}$ which is a rational function, hence continuous on its domain $\mathbb{R} / \{4\}$, thus continuous on $(1, 4), (4, \infty)$.

At $x = 4$

$$\begin{aligned} \lim_{x \rightarrow 4} f(x) &= \lim_{x \rightarrow 2} \frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} \\ &= \lim_{x \rightarrow 2} \frac{x-4}{(x-4)(\sqrt{x}+2)} \\ &= \lim_{x \rightarrow 2} \frac{1}{\sqrt{x}+2} \quad x \neq 4 \\ &= \frac{1}{4} \end{aligned}$$

Thus f has a removable discontinuity at $x = 4$

At $x = 1$

$$1. \quad f(1) = \frac{1}{1+2} = \frac{1}{3}$$

$$2. \quad \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \frac{1}{x+2} \quad x < 0 \\ = \frac{1}{3}$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{\sqrt{x}-2}{x-4} \quad x > 1 \\ = \frac{1}{3}$$

$$\therefore \lim_{x \rightarrow 1} f(x) = \frac{1}{3}$$

$$3. \quad \lim_{x \rightarrow 1} f(x) = \frac{1}{3} = f(1)$$

Hence f is continuous at $x = 1$

Ergo, f is continuous on $(-\infty, -2), (-2, 4), (4, \infty)$ with an infinite discontinuity at $x = -2$ and a removable discontinuity at $x = 4$.

$$\text{k) } f(x) = \begin{cases} \frac{x}{x-1} & x \leq -1 \\ \sqrt{x+1} & -1 < x < 4 \\ \frac{2x-10}{x^2-7x+10} & x \geq 4 \end{cases}$$

On $(-\infty, -1)$, $f(x) = \frac{x}{x-1}$ which is a rational function, hence continuous on its domain $\mathbb{R}/\{1\}$, thus continuous on $(-\infty, -1)$.

On $(-1, 4)$, $f(x) = \sqrt{x+1}$ which is the composition of a root function and a polynomial, thus continuous on its domain $[-1, \infty)$, thus continuous on $(-1, 4)$.

On $(4, \infty)$, $f(x) = \frac{2x-10}{x^2-7x+10}$ which is a rational function, hence continuous on its domain $\mathbb{R}/\{2, 5\}$, thus continuous on $(4, 5)$ and $(5, \infty)$.

At $x = 5$

$$\begin{aligned} \lim_{x \rightarrow 5} f(x) &= \lim_{x \rightarrow 5} \frac{2x-10}{x^2-7x+10} \\ &= \lim_{x \rightarrow 5} \frac{2(x-5)}{(x-2)(x-5)} \\ &= \lim_{x \rightarrow 5} \frac{2}{x-2} \quad x \neq 5 \\ &= \frac{2}{3} \end{aligned}$$

Thus f has a removable discontinuity at $x = 5$

At $x = -1$

$$\begin{aligned} 1. \quad f(-1) &= \frac{-1}{-1-1} = \frac{1}{2} \\ 2. \quad \lim_{x \rightarrow -1^-} f(x) &= \lim_{x \rightarrow -1^-} \frac{x}{x-1} \quad x < -1 & \quad \lim_{x \rightarrow -1^+} f(x) &= \lim_{x \rightarrow -1^+} \sqrt{x+1} \quad x > -1 \\ &= \frac{1}{2} & & = 0 \\ \therefore \lim_{x \rightarrow -1} f(x) &\nexists \end{aligned}$$

Thus f has a jump discontinuity at $x = -1$, but since $\lim_{x \rightarrow -1^-} f(x) = \frac{1}{2} = f(-1)$, then f is continuous from the left at $x = -1$.

At $x = 4$

$$1. \quad f(4) = \frac{2 \cdot 4 - 10}{4^2 - 7 \cdot 4 + 10} = 1$$

$$2. \quad \lim_{x \rightarrow 4^-} f(x) = \lim_{x \rightarrow 4^-} \sqrt{x+1} \quad x < 4 \quad \lim_{x \rightarrow 4^+} f(x) = \lim_{x \rightarrow 4^+} \frac{2x-10}{x^2-7x+10} \quad x > 4$$

$$= \sqrt{5} \quad = 1$$

$$\therefore \lim_{x \rightarrow 4} f(x) \nexists$$

Thus f has a jump discontinuity at $x=4$, but since $\lim_{x \rightarrow 4^+} f(x) = 1 = f(4)$, then f is continuous from the right at $x=4$.

Hence f is continuous on $(-\infty, -1]$, $(-1, 4)$, $[4, 5)$ and $(5, \infty)$ with jump discontinuities at $x = -1, 4$ and a removable discontinuity at $x = 5$.

$$1) \quad f(x) = \begin{cases} \frac{1}{x^2+2x} & x \leq -1 \\ \frac{2}{x^2-x-2} & -1 < x \leq 1 \\ \frac{x^2-3x+2}{x-1} & x > 1 \end{cases}$$

On $(-\infty, -1)$, $f(x) = \frac{1}{x^2+2x}$ which is a rational function, hence continuous on its domain $\mathbb{R}/\{-2, 0\}$, thus continuous on $(-\infty, -2)$ and $(-2, -1)$.

At $x = -2$

$$\lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} \frac{1}{x^2+2x} = \infty \quad \text{case } \frac{b}{0}$$

$$\lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^+} \frac{1}{x^2+2x} = -\infty \quad \text{case } \frac{b}{0}$$

$$\therefore \lim_{x \rightarrow -2} f(x) \nexists$$

Thus f has an infinite discontinuity at $x = -2$

On $(-1, 1)$, $f(x) = \frac{2}{x^2-x-2}$ which is a rational function, hence continuous on its domain $\mathbb{R}/\{-1, 2\}$, thus continuous on $(-1, 1)$.

On $(1, \infty)$, $f(x) = \frac{x^2-3x+2}{x-1}$ which is a rational function, hence continuous on its domain $\mathbb{R}/\{1\}$, thus continuous on $(1, \infty)$.

At $x = -1$

$$1. \quad f(-1) = \frac{1}{(-1)^2+2(-1)} = -1$$

$$2. \quad \lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} \frac{1}{x^2+2x} \quad x < -1$$

$$= -1$$

$$\lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} \frac{2}{x^2 - x - 2} \quad x > -1$$

$$= \infty \quad \text{case } \frac{b}{0}$$

$$\therefore \lim_{x \rightarrow -1} f(x) \nexists$$

Thus f has an infinite discontinuity at $x = -1$, but since

$$\lim_{x \rightarrow -1^-} f(x) = -1 = f(-1), \text{ then } f \text{ is continuous from the left at } x = -1.$$

At $x = 1$

$$1. \quad f(1) = \frac{2}{1^2 - 1 - 2} = -1$$

$$2. \quad \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \frac{2}{x^2 - x - 2} \quad x < 1 \quad \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{x^2 - 3x + 2}{x - 1} \quad x > 1$$

$$= -1 \quad = \lim_{x \rightarrow 1^+} \frac{(x-2)(x-1)}{x-1}$$

$$= \lim_{x \rightarrow 1^+} (x-2)$$

$$= -1$$

$$\therefore \lim_{x \rightarrow 1} f(x) = -1$$

$$3. \quad \lim_{x \rightarrow 1} f(x) = -1 = f(1)$$

Thus f is continuous at $x = 1$.

Ergo, f is continuous on f is continuous on $(-\infty, -2)$, $(-2, -1]$ and $(-1, \infty)$ with infinite discontinuities at $x = -2, -1$.

$$m) \quad f(x) = \begin{cases} \frac{x^2 - 16}{x^2 + 4x} & x < 0 \\ \frac{\sqrt{x+1} - 2}{x^2 - 9} & 0 \leq x \leq 3 \\ \frac{1 - \frac{3}{x}}{x^2 + 2x - 15} & x > 3 \end{cases}$$

On $(-\infty, 0)$, $f(x) = \frac{x^2 - 16}{x^2 + 4x}$ which is a rational function, hence continuous on its domain $\mathbb{R} / \{-4, 0\}$, thus continuous on $(-\infty, -4)$ and $(-4, 0)$.

At $x = -4$

$$\begin{aligned}\lim_{x \rightarrow -4} f(x) &= \lim_{x \rightarrow -4} \frac{x^2 - 16}{x^2 + 4x} \\ &= \lim_{x \rightarrow -4} \frac{(x-4)(x+4)}{x(x+4)} \\ &= \lim_{x \rightarrow -4} \frac{x-4}{x} \\ &= 2\end{aligned}$$

Thus f has a removable discontinuity at $x = -4$

On $(0, 3)$, $f(x) = \frac{\sqrt{x+1}-2}{x^2-9}$ which is a rational function, hence continuous on its domain $\mathbb{R} / \{-3, 3\}$, thus continuous on $(0, 3)$.

On $(3, \infty)$, $f(x) = \frac{1-\frac{3}{x}}{x^2+2x-15}$ which is a rational function, hence continuous on its domain $\mathbb{R} / \{-5, 3\}$, thus continuous on $(3, \infty)$.

At $x = 0$

1. $f(0) = \frac{\sqrt{0+1}-2}{0-9} = \frac{1}{9}$
2. $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow -1^-} \frac{x^2-16}{x^2+4x} \quad x < 0$
 $= \infty$ case $\frac{b}{0}$

$$\begin{aligned}\lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow -1^+} \frac{\sqrt{x+1}-2}{x^2-9} \quad x > 0 \\ &= \frac{1}{9}\end{aligned}$$

$\therefore \lim_{x \rightarrow 0} f(x) \nexists$

Thus f has an infinite discontinuity at $x = 0$, but since $\lim_{x \rightarrow 0^+} f(x) = \frac{1}{9} = f(0)$, then f is continuous from the right at $x = 0$.

At $x = 3$

1. $f(3) \nexists$
2. $\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} \frac{\sqrt{x+1}-2}{x^2-9} \cdot \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} \quad x < 3$
 $= \lim_{x \rightarrow 3^-} \frac{x-3}{(x-3)(x+3)(\sqrt{x+1}+2)}$
 $= \lim_{x \rightarrow 3^-} \frac{1}{(x+3)(\sqrt{x+1}+2)}$
 $= \frac{1}{24}$

$$\begin{aligned}
 \lim_{x \rightarrow 3^+} f(x) &= \lim_{x \rightarrow 3^+} \frac{1 - \frac{3}{x}}{x^2 + 2x - 15} \quad x > 3 \\
 &= \lim_{x \rightarrow 3^+} \frac{x - 3}{x(x+5)(x-3)} \\
 &= \lim_{x \rightarrow 3^+} \frac{1}{x(x+5)} \\
 &= \frac{1}{24} \\
 \therefore \lim_{x \rightarrow 1} f(x) &= \frac{1}{24}
 \end{aligned}$$

Thus f has a removable discontinuity at $x = 3$.

Ergo, f is continuous on $(-\infty, -4)$, $(-4, 0)$, $[0, 3)$ and $(3, \infty)$ with a removable discontinuities at $x = -4$ and $x = 3$, and an infinite discontinuity at $x = 0$.

4. Find all values of a and b that make $f(x)$ continuous on \mathbb{R} , where

$$f(x) = \begin{cases} 3x - a & x \leq -1 \\ x^2 + 1 & -1 < x < 2 \\ b & x \geq 2 \end{cases}$$

On $(-\infty, -1)$, $f(x) = 3x - a$ which is a polynomial function, hence continuous on its domain \mathbb{R} , thus continuous on $(-\infty, -1)$.

On $(-1, 2)$, $f(x) = x^2 + 1$ which is a polynomial function, hence continuous on its domain \mathbb{R} , thus continuous on $(-1, 2)$.

On $(2, \infty)$, $f(x) = b$ which is a polynomial function, hence continuous on its domain \mathbb{R} , thus continuous on $(2, \infty)$.

At $x = -1$

- $f(-1) = -3 - a$

- $$\lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} (3x - a) \quad x < -1 \qquad \lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} (x^2 + 1) \quad -1 < x < 2$$

$$= -3 - a \qquad \qquad \qquad = 2$$

$$\text{If } \lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^+} f(x), \text{ then } -3 - a = 2$$

$$a = -5$$

- $\therefore \lim_{x \rightarrow -1} f(x) = -5 = f(-1)$

Thus f is continuous at $x = -1$.

At $x = 2$

- $f(2) = b$

$$2. \quad \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x^2 + 1) \quad -1 < x < 2$$
$$= 5$$

$$\text{If } \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x), \text{ then } b = 5$$

$$3. \quad \therefore \lim_{x \rightarrow 2} f(x) = 5 = f(2)$$

Thus f is continuous at $x = 2$.

Ergo, f is continuous on \mathbb{R} if $a = -5$ and $b = 5$.

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} b \quad x > 2$$
$$= b$$