MATHEMATICS 201-BNK-05
Advanced Calculus
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XXII – Fourier Series

1. Find the Fourier series for the following functions.

   a) $f(x)=\begin{cases}0 & -5<x<0\\ 4 & 0<x<5\end{cases}$ Period 10

   b) $f(x)=\begin{cases}0 & -\pi<x<0\\ \sin x & 0<x<\pi\end{cases}$ Period 2π

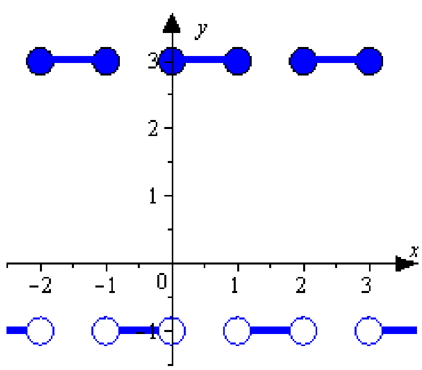
   c) $f(x)=\begin{cases}1 & -2<x<0\\ 3 & 0<x<2\end{cases}$ Period 4

   d) $f(x)=\begin{cases}x+1 & -1<x<0\\ 1-x & 0<x<1\end{cases}$ Period 2

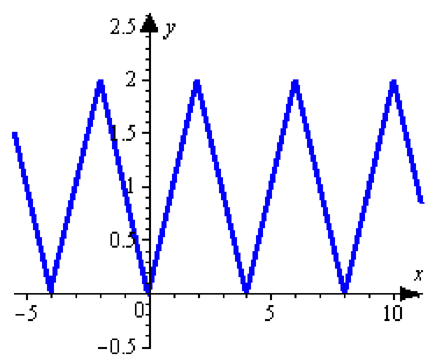
   e) $f(x)=\begin{cases}4 & 0\le x\le 3\\ 0 & 3<x<4\end{cases}$ Period 4

2. For each of the functions whose graph is given below, find the Fourier series and the function $\bar{f}$ to which the Fourier series converges at every point.

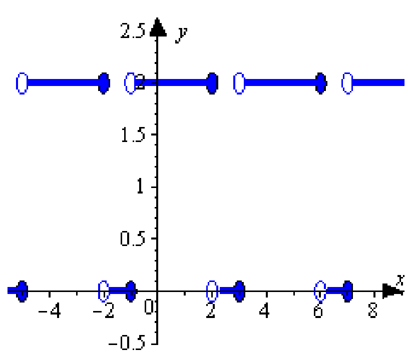
   a)



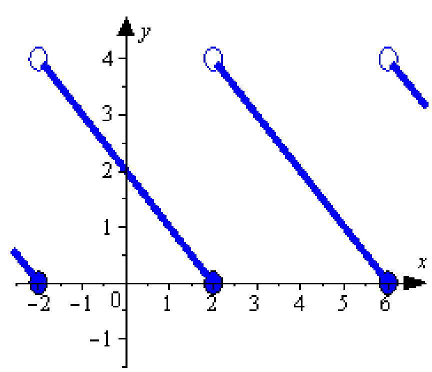
   b)



   c)



   d)



3. Find the Cosine and Sine series for the following function.
   a) $f(x)=5$ $\quad 0<x<10$
   b) $f(x)=x+2$ $\quad 0<x<1$
   c) $f(x)=x^2$ $\quad 0<x<2$
   d) $f(x)=\sin x$ $\quad 0<x<\pi$
   e) $f(x)=x(x-5)$ $\quad 0<x<5$
   f) $f(x)=\begin{cases} x & 0\le x\le 2 \\ 4-x & 2<x<4 \end{cases}$ $\quad 0<x<4$

4. Using Parseval's equality and the results of question 3d, prove that

$$\frac{1}{1^2\cdot 3^2}+\frac{1}{3^2\cdot 5^2}+\frac{1}{5^2\cdot 7^2}+\cdots=\frac{\pi^2-8}{16}$$

5. Consider the function $f(x)=\begin{cases} -1 & -1<x<0 \\ 1 & 0<x<1 \end{cases}$ periodic with a period of 2.
   a) Find the Fourier series for $f(x)$
   b) Use Parseval's equality to prove that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$.

6. Use integration on the results of question 3c to find the Fourier series for $f(x)=x^3-4x$ on $(-2,2)$.

7. Use differentiation on the results of question 1b to find the Fourier series for $f(x)=\begin{cases} 0 & -\pi<x<0 \\ \cos x & 0<x<\pi \end{cases}$ on $(-2,2)$.

ANSWERS

1. a) $\bar{f}(x)=2+\sum_{k=1}^{\infty}\frac{8}{(2k-1)\pi}\sin\left(\frac{(2k-1)\pi x}{5}\right)$

   b) $\bar{f}(x)=\frac{1}{\pi}+\frac{1}{2}\sin x-\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{2}{4k^2-1}\cos(2kx)$

   c) $\bar{f}(x)=2+\sum_{k=1}^{\infty}\frac{4}{\pi(2k-1)}\sin\frac{\pi(2k-1)x}{2}$

   d) $\bar{f}(x)=\frac{1}{2}+\sum_{k=1}^{\infty}\frac{4}{\pi^2(2k-1)^2}\cos(\pi(2k-1)x)$

   e) $\bar{f}(x)=3+\sum_{k=1}^{\infty}\frac{4(-1)^k}{(2k-1)\pi}\cos\frac{(2k-1)\pi x}{2}+\sum_{k=1}^{\infty}\frac{4}{(2k-1)\pi}\sin\frac{(2k-1)\pi x}{2}+\sum_{k=1}^{\infty}\frac{4}{(2k-1)\pi}\sin((2k-1)kx)$

2. a) $\bar{f}(x)=1+\sum_{k=1}^{\infty}\frac{8}{(2k-1)\pi}\sin((2k-1)\pi x)$

   $$\bar{f}(x)=\begin{cases}-1 & -1<x<0\\ 1 & x=0\\ 3 & 0<x<1\\ 1 & x=1\end{cases}\quad \text{period } 2$$

   b) $\bar{f}(x)=1-\sum_{k=1}^{\infty}\frac{8}{(2k-1)^2\pi^2}\cos\frac{(2k-1)\pi x}{2}$

   $\bar{f}(x)=|x| \quad -2<x\le 2 \quad$ Period 4

   c) $\bar{f}(x)=\frac{3}{2}+\sum_{k=1}^{\infty}\frac{2(-1)^{k+1}}{(2k-1)\pi}\cos\frac{(2k-1)\pi x}{2}+\sum_{k=1}^{\infty}\frac{2}{(2k-1)\pi}\sin\frac{(2k-1)\pi x}{2}$

   $-\sum_{k=1}^{\infty}\frac{2}{(2k-1)\pi}\sin((2k-1)kx)$

   $$\bar{f}(x)=\begin{cases}0 & -2<x<-1\\ 1 & x=-1\\ 2 & -1<x<2\\ 1 & x=2\end{cases}\quad \text{Period } 4$$

   d) $\bar{f}(x)=2+\sum_{k=1}^{\infty}\frac{4(-1)^k}{\pi k}\sin\frac{k\pi x}{2}$

   $$\bar{f}(x)=\begin{cases}2-x & -2<x<2\\ 2 & x=2\end{cases}\quad \text{Period } 4$$

3. a) Cosine: $\bar{f}(x)=5$

   Sine: $\bar{f}(x)=\sum_{k=1}^{\infty}\frac{20}{(2k-1)\pi}\sin\frac{(2k-1)\pi x}{10}$

b) Cosine: $\overline{f}(x)=\frac{5}{2}-\sum_{k=1}^{\infty}\frac{4}{(2k-1)^2\pi^2}\cos((2k-1)\pi x)$

Sine: $\overline{f}(x)=\sum_{k=1}^{\infty}\left(\frac{10}{(2k-1)\pi}\sin((2k-1)\pi x)-\frac{1}{k\pi}\sin(2k\pi x)\right)$

c) Cosine: $\overline{f}(x)=\frac{4}{3}+\sum_{k=1}^{\infty}\frac{16(-1)^k}{\pi^2k^2}\cos\frac{k\pi x}{2}$

Sine: $\overline{f}(x)=-\sum_{k=1}^{\infty}\left(\frac{8\left(4-(2k-1)^2\pi^2\right)}{\pi^3(2k-1)^3}\sin\frac{(2k-1)\pi x}{2}+\frac{4}{k}\sin(k\pi x)\right)$

d) Cosine: $\overline{f}(x)=\frac{2}{\pi}-\sum_{k=1}^{\infty}\frac{4}{(4k^2-1)\pi}\cos(2kx)$

Sine: $\overline{f}(x)=\sin x$

e) Cosine: $\overline{f}(x)=\frac{-25}{6}+\sum_{k=1}^{\infty}\frac{25}{k^2\pi^2}\cos\frac{2k\pi x}{5}$

Sine: $\overline{f}(x)=-\sum_{k=1}^{\infty}\frac{200}{(2k-1)^3\pi^3}\sin\frac{(2k-1)\pi x}{5}$

f) Cosine: $\overline{f}(x)=1-\sum_{k=1}^{\infty}\frac{8}{(2k-1)^2\pi^2}\cos\frac{(2k-1)\pi x}{2}$

Sine: $\overline{f}(x)=\sum_{k=1}^{\infty}\frac{16(-1)^{k+1}}{(2k-1)^2\pi^2}\sin\frac{(2k-1)\pi x}{4}$

5. a) $\overline{f}(x)=\sum_{k=1}^{\infty}\frac{4}{(2k-1)\pi}\sin((2k-1)\pi x)$

6. $\overline{f}(x)=\sum_{k=1}^{\infty}\frac{96(-1)^k}{k^3\pi^3}\sin\frac{k\pi x}{2}$

7. $\overline{f}(x)=\frac{1}{2}\cos x+\sum_{k=1}^{\infty}\frac{4k}{\pi(4k^2-1)}\sin 2kx$