XXII – Applications of Differential Equations

1. Suppose that the rate at which a population of size $y(t)$ at time t changes is proportional to the amount present. This gives rise to the differential equation:

$$\begin{cases} \dfrac{dy}{dt} = ky \\ y(0) = y_0 \end{cases}$$

where y_0 is the initial population. This is known as the ***Malthus model***.

   a) Solve the Malthus model.
   b) For which values of k does the population increase?
   c) Evaluate $\lim_{t\to\infty} y(t)$ when $k > 0$.

2. Most populations do not follow the Malthus model. In a restricted environment and with limited food supply, the population cannot exceed a maximal size M at which it consumes its entire food supply. Supposing that the rate of growth of the population is proportional to its size and the amount by which it falls short of its maximal size, this gives us the logistic equation,

$$\begin{cases} \dfrac{dy}{dt} = ky(M - y) \\ y(0) = y_0 \end{cases}$$

   a) Solve the logistic equation.
   b) Evaluate $\lim_{t\to\infty} y(t)$ (assuming $k > 0$).

3. **Newton's law of cooling** states that the rate at which the temperature $T(t)$ changes in a cooling body is proportional to the difference between the temperature of the body and the constant temperature T_S of the surrounding medium.

   a) Set up a differential equation modeling the temperature of a cooling body.
   b) Solve this equation.
   c) A hot cup of coffee is initially at 100°C when poured. How long does it take for the coffee to reach a temperature of 50°C if it is 80°C after 15 minutes and the room temperature is 30°C?

4. The simple electric circuit contains an electromotive force (a battery or generator) that produces a voltage of $E(t)$ volts (V) and a current of $I(t)$ amperes (A) at time t. The circuit contains a resistor with a resistance of R ohms (Ω) and an inductor with an inductance of L henries (H). By Ohm's Law, the voltage drop due to the resistor is RI. The voltage drop due to the inductor is $L\frac{dI}{dt}$. So by Kirchhoff's Law we have

$$L\frac{dI}{dt} + RI = E$$

Suppose that in the simple circuit, the resistance is 12 Ω, the inductance is 4 H, and a battery gives a constant voltage of 60 V and the switch is closed when $t = 0$ so the current starts with $I(0) = 0$.

   a) Find $I(t)$.
   b) Find the current after one second.
   c) Find the limiting value of the current, given by $\lim_{t\to\infty} I(t)$.

5. (**Escape Velocity**) Suppose that a rocket is launched from the earth's surface. At a great (radial) distance r from the center of the earth, the rocket's acceleration is not the constant g. Instead, according to Newton's law of gravitation, $a = \frac{k}{r^2}$, where k is the constant of proportionality ($k > 0$ if the rocket is falling towards the earth; $k < 0$ if the rocket is moving away from the earth). This gives rise to the differential equation

$$\begin{cases} \dfrac{dv}{dt} = \dfrac{-gR^2}{r^2} \\ v(R) = v_0 \end{cases}$$

   a) Show that $\frac{dv}{dt} = v\frac{dv}{dr}$.
   b) Solve the differential equation.
   c) Find $\lim_{r\to\infty} v^2$. If $v > 0$ (so that the rocket does not fall to the ground), show that the minimum value of v_0 for which this is true (even for large values of r) is $v_0 = \sqrt{2gR}$. This value is called the **escape velocity** and signifies the minimum velocity required so that the rocket does not return to the earth.
   d) Calculate the escape velocity on earth using $g = 9.8\text{ m/s}^2$ and $R = 6376\text{ km}$.

6. Atomic waste is placed in sealed canisters and dumped in the ocean. It has been determined that the seal will not break and leak the waste when the canister hits the bottom of the ocean as long as the velocity of the canister is less than 12 m/s when it hits the bottom. Using Newton's second law, it can be shown that the velocity must satisfy

$$\begin{cases} m\dfrac{dv}{dt} = W - B - kv \\ v(0) = 0 \end{cases}$$

where *W* is the weight of the canister, *B* the buoyancy force, and –*kv* the drag force.

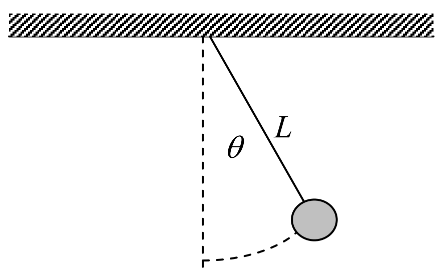
   a) Solve this equation.
   b) If $W = 2254$ Newtons and $B = 2090$ Newtons, $k = 0.637$ and $m = \frac{W}{g} = 230$ kg, determine the time at which the velocity is 12 m/s.
   c) Determine the depth *H* of the ocean so that the seal will not break when the canister hits the bottom.

7. According to the **Law of Mass Action**, if the temperature is constant, then the velocity of a chemical reaction is proportional to the product of the concentration of the substances that are reacting. The reaction $A + B \to M$ combines *a* moles per liter of substance *A* and *b* moles per liter of substance *B*. If $y(t)$ is the number of moles per liter that have reacted after time *t*, the reaction rate is given by the differential equation

$$\begin{cases} \dfrac{dy}{dt} = k(a-y)(b-y) \\ y(0) = 0 \end{cases}$$

   a) Solve for $y(t)$.
   b) Find $\lim\limits_{t\to\infty} y(t)$ if $a > b$ and if $b > a$, assuming that $k > 0$.

8. Suppose that a mass *m* is attached to the end of a rod of length *L*, the weight of which is negligible. Let θ be the angle (in radians) of displacement as a function of time *t*.



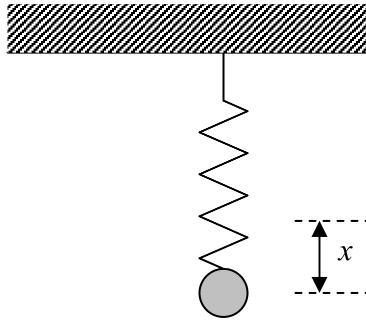
For small values of θ, this gives rise to the differential equation

$$\begin{cases} \dfrac{d^2\theta}{dt^2} + \dfrac{g}{L}\theta = 0 \\ \theta(0) = \theta_0,\ \frac{d\theta}{dt}(0) = v_0 \end{cases}$$

where *g* is the acceleration due to gravity.

   a) Solve this equation assuming $L = 2$ m, $\theta_0 = \frac{\pi}{6}$ and $v_0 = 0$ rad/s.
   b) Solve this equation assuming $L = 2$ m, $\theta_0 = 0$ and $v_0 = 2$ rad/s.

9. Suppose that an object of mass m is attached to an elastic spring that is suspended from the ceiling.



According to Hooke's law and Newton's Second law of motion, the displacement x of the object with respect to the equilibrium position at time t, is found by

$$\begin{cases} m\dfrac{d^2x}{dt^2} + kx = 0 \\ x(0) = \alpha, \quad \frac{dx}{dt}(0) = \beta \end{cases}$$

where α represents the initial position and β the initial velocity. Solve this system assuming that an object has a mass $m = 1 \text{ kg}$, the spring a constant $k = 64 \text{ kg/m}$, and the object is releasd from rest 1 m below the original position.

10. Suppose that in the previous problem, there is a resistive force on the spring due to damping. The differential equation then becomes:

$$\begin{cases} m\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + kx = 0 \\ x(0) = \alpha, \quad \frac{dx}{dt}(0) = \beta \end{cases}$$

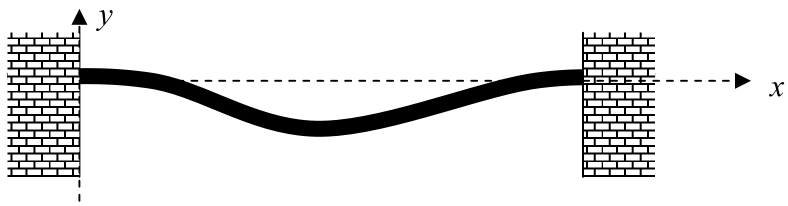
where $c > 0$ is a constant. Solve this problem using the same initial conditions as in question 8 and with the following constants

a) $m = 1 \text{ kg}$, $k = 4 \text{ kg/m}$, and $c = 5$. (This system is said to be **overdamped**)

b) $m = 1 \text{ kg}$, $k = 4 \text{ kg/m}$, and $c = 4$. (This system is said to be **critically damped**)

c) $m = 1 \text{ kg}$, $k = 16 \text{ kg/m}$, and $c = 2$. (This system is said to be **underdamped**)

11. A beam of length $l = 1$ m is fixed between two walls. Suppose the shape of the beam when it is deflected (under its weight) is given by the graph of $y(x)$ where x is the distance from the left end of the bean.

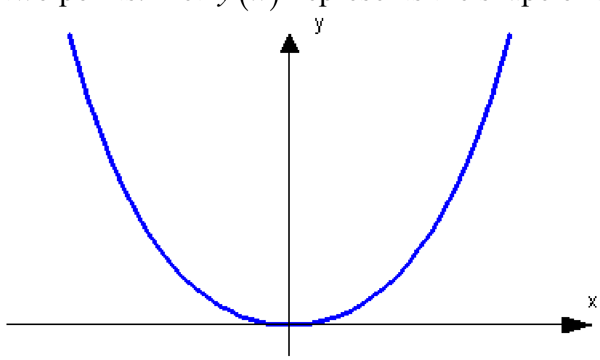


This gives rise to the differential equation

$$\begin{cases} \dfrac{d^4y}{dx^4} = -w \\ y(0) = 0,\ \frac{dy}{dx}(0) = 0 & \text{(The left side is fixed)} \\ y(1) = 0,\ \frac{dy}{dx}(1) = 0 & \text{(The right side is fixed)} \end{cases}$$

where w is a constant representing the weight distribution of the beam (which would not be constant if the re were charges on the beam). Solve this equation, assuming $w = 48$.

12. A cable hangs between two points. Let $y(x)$ represents the shape of the cable,



then it can be shown that is satisfies the differential equation

$$\begin{cases} \dfrac{d^2y}{dx^2} = k\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \\ y(0) = 0,\ \frac{dy}{dx}(0) = 0 \end{cases}$$

where k is a constant depending on the weight of the cable. Solve this equation, assuming $k = 1$.

ANSWERS

1. a) $y = y_0 e^{kt}$ b) when $k > 0$ c) $\lim_{t\to\infty} y(t) = \infty$
2. a) $y = \frac{y_0 M}{y_0 + (M - y_0)e^{-kMt}}$ b) $\lim_{t\to\infty} y(t) = M$
3. a) $\frac{dT}{dt} = k(T - T_S)$, $T(0) = T_0$ b) $T(t) = (T_0 - T_S)e^{kt} + T_S$
   c) $t \approx 55.85$ min
4. a) $I(t) = 5(1 - e^{-3t})$ b) $I(1) \approx 4.75$ A c) $\lim_{t\to\infty} I(t) = 5$
5. b) $v^2 = \frac{2gR^2}{r} + v_0^2 - 2gR$ c) $\lim_{r\to\infty} r^2 = v_0^2 - 2gR$ d) 671 km/h
6. a) $v(t) = \frac{W-B}{k}\left(1 - e^{-\frac{kt}{m}}\right)$ b) 17.23 sec c) $H = 104.18$
7. a) $y(t) = \frac{ab\left(e^{k(a-b)t} - 1\right)}{ae^{k(a-b)t} - b}$
   b) If $a > b$ then $\lim_{t\to\infty} y(t) = b$, and if $a < b$ then $\lim_{t\to\infty} y(t) = a$
8. a) $\theta(t) = \frac{\pi}{6}\cos\left(\sqrt{4.9}t\right)$ b) $\theta(t) = \frac{2}{\sqrt{4.9}}\sin\left(\sqrt{4.9}t\right)$
9. $x(t) = -\cos(8t)$
10. a) $x(t) = \frac{1}{3}e^{-4t} - \frac{4}{3}e^{-t}$ b) $x(t) = -e^{-2t} - 2te^{-2t}$
    c) $x(t) = -e^{-t}\left(\cos\sqrt{15}t + \frac{\sqrt{15}}{15}\sin\sqrt{15}t\right)$
11. $y(x) = -2x^4 + 4x^3 - 2x^2$
12. $y = \frac{e^x + e^{-x}}{2} - 1$