MATHEMATICS 201-BNK-05
Advanced Calculus
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XXI – Higher Order Differential Equations

1. Solve the following higher order differential equations.

a) $\dfrac{d^2y}{dx^2} = e^x + e^{-x}$

b) $\dfrac{d^6y}{dx^6} = 1$

c) $\dfrac{d^2y}{dx^2} = \sec^2(3x)$

d) $\dfrac{d^4y}{dx^4} = x^2 + 3\ln x$

2. Solve the following linear differential equations with constant coefficients.

a) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - 15y = 0$

b) $2\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} - 3y = 0$

c) $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 0$

d) $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 34y = 0$

e) $\dfrac{d^2y}{dx^2} + 36y = 0$

f) $\dfrac{d^2y}{dx^2} - 36y = 0$

g) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 0$

h) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0$

i) $\dfrac{d^3y}{dx^3} - 2\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} = 0$

j) $\dfrac{d^4y}{dx^4} + 4\dfrac{d^2y}{dx^2} = 0$

k) $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = 0$

l) $\dfrac{d^4y}{dx^4} - 16y = 0$

m) $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 4y = 3$

n) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 5y = 5x^2 + 2x + 1$

o) $\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 15y = xe^x$

p) $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = e^{2x} + x$

q) $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 5y = \sin(2x) + \cos(2x)$

r) $\dfrac{d^4y}{dx^4} + 5\dfrac{d^2y}{dx^2} + 4y = \cos(3x+2)$

s) $\dfrac{d^3y}{dx^3} + \dfrac{dy}{dx} = \sin x + \cos(2x)$

t) $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = xe^x$

u) $\dfrac{d^4y}{dx^4} + 8\dfrac{d^2y}{dx^2} + 16y = x^2$

v) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = xe^{3x} + 4e^x\cos(2x)$

w) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 17x\sin(2x)$

x) $\dfrac{d^4y}{dx^4} + \dfrac{d^2y}{dx^2} = x^2\sin x + x$

3. Solve the following differential equations using an appropriate substitution.

a) $\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2=1$

b) $\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}=\sec x$

c) $x\frac{d^2y}{dx^2}-\frac{dy}{dx}=\frac{-2}{x}-\ln x$

d) $x\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}=0$

e) $\frac{d^2y}{dx^2}=\frac{1}{y^3}$

f) $x\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^3-\frac{dy}{dx}=0$

g) $y\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2=y^2\frac{dy}{dx}$

h) $y\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2=4y\left(\frac{dy}{dx}\right)^3$

i) $\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^3+\frac{dy}{dx}$

j) $y\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2=y^2\ln y$

4. The second-order linear differential equation

$$x^2\frac{d^2y}{dx^2}+ax\frac{dy}{dx}+by=F(x)$$

where a and b are constants is called the Cauchy-Euler equation. Because of the x^2- and x-factors, it does not have constant coefficients, so the method seen in class will not work. However, if we make the change of variables $x=e^z$, then the resultant equation will be linear with constant coefficients.

a) Show that with the transformation $x=e^z$, we have

$$x\frac{dy}{dx}=\frac{dy}{dz} \qquad x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}$$

b) Solve the equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+9y=18$, $x>0$.

c) Solve the equation $\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}=0$, $r>0$.

5. Solve the following differential equations.

a) $x\frac{d^2y}{dx^2}+2\frac{dy}{dx}+x=1$ $\quad y(4)=2$, $y'(1)=3$

b) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}-8y=x+e^{2x}$ $\quad y(0)=1$, $y'(0)=2$

c) $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3y=0$ $\quad y(4)=0$, $\left.\frac{dy}{dx}\right|_{y=2}=-1$

d) $\frac{d^3y}{dx^3}+x=\cos x$ $\quad y(0)=2$, $y'(\pi)=3$, $y''\left(\frac{\pi}{4}\right)=1$

e) $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+34y=\sin(5x)$ $\quad y(0)=0$, $y'(0)=1$

f) $\frac{d^5y}{dx^5}-8\frac{d^2y}{dx^3}=96x$ $\quad y(0)=0$, $y'(0)=0$, $y''(0)=0$, $y'''(0)=0$, $y^{(iv)}(0)=0$

ANSWERS

1. a) $y = e^x + e^{-x} + K_1 x + K_2$
   b) $y = \frac{1}{720}x^6 + K_1x^5 + K_2x^4 + K_3x^3 + K_4x^2 + K_5x + K_6$
   c) $y = \frac{1}{9}\ln(\sec 3x) + K_1x + K_2$
   d) $y = \frac{1}{360}x^6 + \frac{1}{8}x^4 \ln x - \frac{25}{96}x^4 + K_1x^3 + K_2x^2 + K_3x + K_4$
2. a) $y = K_1e^{3x} + K_2e^{-5x}$ b) $y = K_1e^{\frac{-1}{2}x} + K_2e^{3x}$ c) $y = (K_1 + K_2x)e^{3x}$
   d) $y = e^{3x}(K_1\cos 5x + K_2 \sin 5x)$ e) $y = K_1\cos 6x + K_2\sin 6x$
   f) $y = K_1e^{6x} + K_2e^{-6x}$ g) $y = (K_1 + K_2x)e^{-2x}$
   h) $y = e^{\frac{-1}{2}x}\left(K_1\cos\left(\frac{\sqrt{3}}{2}x\right) + K_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right)$ i) $y = K_1 + K_2e^{-x} + K_3e^{3x}$
   j) $y = K_1e^{2x} + K_2e^{-2x} + K_3\cos 2x + K_4\sin 2x$ k) $y = K_1e^x + K_2e^{2x} + K_3e^{3x}$
   l) $y = K_1e^{2x} + K_2e^{-2x} + K_3\cos 2x + K_4\sin 2x$ m) $y = K_1e^x + K_2e^{4x} + \frac{3}{4}$
   n) $y = K_1e^{2x}\sin x + K_2e^{2x}\cos x + x^2 + 2x + \frac{7}{5}$ o) $y = K_1e^{3x} + K_2e^{5x} + \left(\frac{3}{32} + \frac{1}{8}x\right)e^x$
   p) $y = K_1e^{2x} + K_2e^{3x} - xe^{2x} + \frac{1}{6}x + \frac{5}{36}$
   q) $y = K_1e^{-5x} + K_2e^{-x} + \frac{13}{145}\sin(2x) - \frac{11}{145}\cos(2x)$
   r) $y = K_1\cos x + K_2\sin x + K_3\cos(2x) + K_4\sin(2x) + \frac{1}{40}\cos(3x+2)$
   s) $y = K_1\cos x + K_2\sin x - \frac{1}{2}x\sin x - \frac{1}{6}\sin 2x + K_3$
   t) $y = K_1e^x + K_2xe^x + K_3x^2e^x + \frac{1}{24}x^4e^x$
   u) $y = K_1\sin 2x + K_2\cos 2x + K_3x\sin 2x + K_4x\cos 2x + \frac{1}{16}x^2 - \frac{1}{16}$
   v) $y = K_1e^x + K_2e^{-2x} + \frac{130x-91}{1300}e^{3x} + e^x\left(\frac{6}{13}\sin 2x - \frac{4}{13}\cos 2x\right)$
   w) $y = K_1e^{-x}\sin(2x) + K_2e^{-x}\cos(2x) + \left(x - \frac{2}{17}\right)\sin 2x + \left(-4x + \frac{76}{17}\right)\cos 2x$
   x) $y = K_1 + K_2x + \frac{1}{6}x^3 + K_3\sin x + K_4\cos x - \frac{17}{4}x\cos x - \frac{5}{4}x^2\sin x + \frac{1}{6}x^3\cos x$
3. a) $y = \ln\left|\sec(x + K_1)\right| + K_2$ b) $y = -\cos x + K_1\sin x + K_2$
   c) $y = K_1x^2 + K_2 + (x+1)\ln x$ d) $y = K_1x^5 + K_2x + K_3$
   e) $K_1y^2 = (K_1x + K_2)^2 + 1$ f) $x^2 + y^2 = K_1y + K_2$
   g) $\frac{y}{y+K_1} = K_2e^{k_1x}$ h) $K_1\ln y - y^2 = x + K_2$
   i) $y = \arcsin(K_2e^x) + K_1$ j) $\ln y = K_1e^x + K_2e^{-x}$
4. b) $y = K_1\sin(3\ln x) + K_2\cos(3\ln x) + 2$ c) $u = \frac{K_1}{r} + K_2r$
5. a) $y = \frac{-1}{6}x^2 + \frac{1}{2}x - \frac{17}{6x} + \frac{27}{8}$ b) $y = \frac{5}{288}e^{-4x} + \frac{73}{72}e^{2x} + \frac{1}{6}xe^{2x} - \frac{1}{8}x - \frac{1}{32}$
   c) $y^3 - 18y = 6x - 24$
   d) $y = -\sin x - \frac{1}{24}x^4 + \left(\frac{1}{2} - \frac{\sqrt{2}}{4} + \frac{\pi^2}{64}\right)x^2 + \left(2 + \frac{13}{96}\pi^3 - \pi + \frac{\sqrt{2}\pi}{2}\right)x + 2$
   e) $y = e^{3x}\left(\frac{-10}{327}\cos 5x + \frac{114}{545}\sin 5x\right) + \frac{1}{109}\sin 5x + \frac{10}{327}\cos 5x$
   f) $y = -2x^3 + \frac{1}{2}e^{2x} + e^{-x}\cos\left(\sqrt{3}x\right) - \frac{3}{2}$