



MATHEMATICS 201-BNK-05

Advanced Calculus

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Winter 2011

XVIII - Triple Integrals in Cylindrical and Spherical Coordinates

1. Sketch the solid whose volume is given by the integral and evaluate the integral.

a) $\int_0^3 \int_0^{2\pi} \int_r^3 r dz d\theta dr$

b) $\int_0^{\frac{\pi}{4}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^4 \rho^2 \sin \phi d \rho d \theta d \phi$

c) $\int_0^{\pi} \int_0^{2\sin\theta} \int_{\frac{1}{2}r^2}^{4-r} r dz dr d\theta$

d) $\int_0^{\pi} \int_0^{\frac{\pi}{4}} \int_{\sec\phi}^{2\cos\phi} \rho^2 \sin \phi d \rho d \phi d \theta$

2. Use cylindrical coordinates to find the volume of the solid S .

a) S is enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 16$.

b) S lies inside the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 9$.

c) S is bounded by cone $x^2 + y^2 = z^2$ and the paraboloid $z = 5 - 4x^2 - 4y^2$.

d) S is the solid between the cylinders $x^2 + y^2 = x$ and $x^2 + y^2 = 2x$, and bounded below by $z = 0$ and above by $y + z = 3$.

3. Use spherical coordinates to find the volume of the solid S

a) S is enclosed by the sphere $x^2 + y^2 + z^2 = 16$ and the cone $z = \sqrt{3x^2 + 3y^2}$.

b) S is the wedge cut from a sphere of radius a by two planes that intersect along a diameter at an angle of $\frac{\pi}{4}$.

c) S is inside the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and outside the sphere $x^2 + y^2 + z^2 = z$.

4. Evaluate the following integrals using either Cartesian, cylindrical or spherical coordinates, whichever seems more appropriate.

a) $\int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{x^2+y^2}^9 x dz dy dx$

b) $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} e^{-(x^2+y^2+z^2)^{\frac{3}{2}}} dz dy dx$

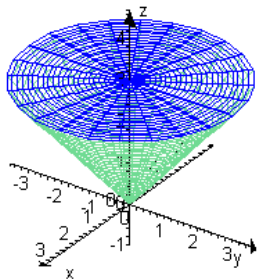
c) $\iiint_S \sqrt{x^2 + y^2} dV$ where S is the region that lies inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -4$ and $z = 5$.

d) $\iiint_S (x^2 + y^2 + z^2) dV$ where S is the region in the first octant bounded by the sphere $x^2 + y^2 + z^2 = 25$, the cone $z = \sqrt{x^2 + y^2}$ and the cone $z = 2\sqrt{x^2 + y^2}$.

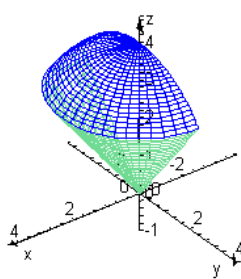
- e) $\iiint_S x dV$ where S is enclosed by the planes $z = 0$, $x + y - z = -3$ and by the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.
- f) $\iiint_S \sin \sqrt{x^2 + y^2 + z^2} dV$ where S is the solid within the sphere $x^2 + y^2 + z^2 = 49$, above the xy -plane and below the cone $z = \sqrt{x^2 + y^2}$.
5. Find the centroid of S .
- a) S is the solid inside the sphere $x^2 + y^2 + z^2 = 2$ and the cone $z = \sqrt{x^2 + y^2}$.
- b) S is the solid bounded by the paraboloid $z = 4x^2 + 4y^2$ and the plane $z = a$, $a > 0$.
- c) S is bounded by the xy -plane and the hemispheres $y = \sqrt{9 - x^2 - z^2}$ and $y = \sqrt{16 - x^2 - z^2}$.
6. Find the center of gravity for the solid S .
- a) S is enclosed by the sphere $x^2 + y^2 + z^2 = 4z$ and the density is $\delta(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.
- b) S is the solid that is bounded by the cylinder $x^2 + y^2 = 1$, the cone $z = \sqrt{x^2 + y^2}$, and the xy -plane if the density is $\delta(x, y, z) = z$.
- c) S is bounded above by the plane $z = y$ and below by the paraboloid $z = x^2 + y^2$ if the density is $\delta(x, y, z) = \sqrt{x^2 + y^2}$.
7. Find the moments of inertia for the hollow cylinder $a_1^2 \leq x^2 + y^2 \leq a_2^2$, $0 \leq z \leq h$, assuming a constant density $\delta(x, y, z) = \delta$.
8. Find the moments of inertia for the solid lying between two concentric hemispheres of radii r and R , where $r < R$, assuming a constant density $\delta(x, y, z) = \delta$.
9. Evaluate the improper integrals, if they converge.
- a) $\iiint_{\mathbb{R}^3} e^{-\sqrt{x^2 + y^2 + z^2}} dV$
- b) $\iiint_S \frac{1}{1 + x^2 + y^2 + z^2} dV$ where S is the inside of the cone $z = \sqrt{x^2 + y^2}$
- c) $\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^\infty \frac{1}{1 + x^2 + y^2 + z^2} dz dy dx$

ANSWERS

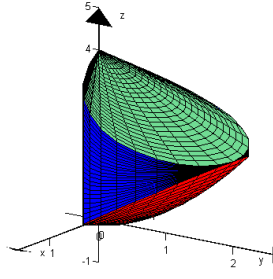
1. a) 9π



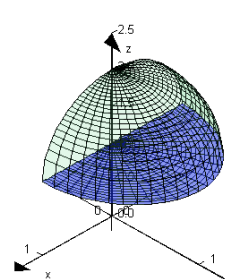
b) $\frac{64\pi}{3} - \frac{32\sqrt{2}\pi}{3}$



c) $\frac{13\pi}{4} - \frac{32}{9}$



d) $\frac{\pi}{3}$



2. a) 128π

b) $36\pi - \frac{64\sqrt{2}\pi}{3}$

c) $\frac{7\pi}{3}$

d) $\frac{9\pi}{4}$

3. a) $\frac{128\pi}{3} - \frac{64\sqrt{3}\pi}{3}$

b) $\frac{\pi a^3}{6}$

c) $\frac{31\pi}{6}$

4. a) 0

b) $\frac{\pi}{3} - \frac{\pi}{3e}$

c) 384π

d) $500\sqrt{5}\pi - 625\sqrt{2}\pi$

e) $\frac{65\pi}{4}$

f) $\sqrt{2}\pi(14\sin(7) - 47\cos(7) - 2)$

5. a) $\left(0, 0, \frac{3(\sqrt{2}+1)}{8}\right)$

b) $\left(0, 0, \frac{2a}{3}\right)$

c) $\left(0, \frac{525}{296}, \frac{525}{296}\right)$

6. a) $\left(0, 0, \frac{16}{7}\right)$

b) $\left(0, 0, \frac{8\pi}{15}\right)$

c) $\left(0, \frac{4}{7}, \frac{24}{49}\right)$

7. $I_x = \frac{\pi\delta h}{4}(a_2^4 - a_1^4) + \frac{\pi\delta h^3}{3}(a_2^2 - a_1^2)$, $I_y = \frac{\pi\delta h}{4}(a_2^4 - a_1^4) + \frac{\pi\delta h^3}{3}(a_2^2 - a_1^2)$, $I_z = \frac{\pi\delta h}{2}(a_2^4 - a_1^4)$

8. $I_x = \frac{8\pi\delta}{15}(R^5 - r^5)$, $I_y = \frac{8\pi\delta}{15}(R^5 - r^5)$, $I_z = \frac{8\pi\delta}{15}(R^5 - r^5)$

9. a) 8π

b) Diverges

c) $\frac{\pi^2}{2}(\sqrt{5} - 1)$