MATHEMATICS 201-BNK-05
Advanced Calculus
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XV - Applications of Double Integrals

1. Find the center of gravity for the that occupies the region D and has density function δ.
   a) $R=\{(x,y)\mid 0\le x\le 2, 0\le y\le 3\}$; $\delta(x,y)=x^2+y^2$
   b) R is the triangular region with vertices $(0,0)$, $(2,1)$ and $(0,3)$; $\delta(x,y)=x+y$
   c) R is the region in the first quadrant bounded by the curves $y=x+x^2$, $y=0$, $x=0$ and $x=2$; $\delta(x,y)=\dfrac{y}{1+x}$.
   d) R is the region bounded by $y=e^x$, $y=0$, $x=0$ and $x=1$; $\delta(x,y)=y$
   e) R is the region bounded by the x-axis and the upper half of the circle $x^2+y^2=1$; $\delta(x,y)=\sqrt{x^2+y^2}$
   f) R is the region enclosed by the leaf of the rose $r=\sin 2\theta$ in the first quadrant; $\delta(x,y)=\sqrt{x^2+y^2}$.
   g) R is the region bounded by $y=\sin\frac{\pi x}{L}$, $y=0$, $x=0$ and $x=L$; $\delta(x,y)=ky$

2. Find the centroid of the region R.
   a) R is bounded by the curves $y^2=2x$ and $y=x$
   b) R is bounded by the curves $y=\frac{1}{4}x^2$, $xy=2$ and $3y-2x=4$, $x\ge 1$, $y\ge 1$
   c) R is the region in the first quadrant between the circles $x^2+y^2=a^2$ and $x^2+y^2=b^2$, $(a<b)$
   d) R is the region inside the cardioid $r=2(1+\cos\theta)$ and outside the circle $r=2$.
   e) R is the leaf of $r=4\cos 2\theta$ that cuts the positive x-axis.
   f) R is the region bounded by $y=\frac{1}{x^3}$, $y=0$ and $x=1$, with $x\ge 1$.

3. Find the moment of Inertia I_x, I_y and I_0 for the given lamina.
   a) R is the region bounded by $y=e^x$, $y=0$, $x=0$ and $x=1$; $\delta(x,y)=y$.
   b) R is the region enclosed by the leaf of the rose $r=\sin 2\theta$ in the first quadrant; $\delta=\sqrt{x^2+y^2}$.
   c) R is the same infinite region as in question 2(f) and $\delta(x,y)=1$.

4. The average value of a function $f(x,y)$ over a region R is defined to be $f_{avg} = \frac{1}{A(R)} \iint_A f(x,y)\,dA$ where $A(R)$ is the area of R. Find the average value of the given functions over the region R.
   a) $f(x,y) = x^2 - xy$ where R is the region enclosed by $y = x$ and $y = 3x - x^2$.
   b) $f(x,y) = \frac{1}{x+y+1}$ where R is the triangle with vertices $(0,0)$, $(0,a)$ and $(a,0)$
   c) $f(x,y) = e^{-x^2-y^2}$ where R is the region between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ (where $a > 0$ and $b > a$).
   d) $f(x,y) = x$ where R is the region between $y = \frac{1}{1+x^2}$ and the x-axis, with $x \geq 0$.
   e) $f(x,y) = xy$ where R is the region between $y = \frac{1}{1+x^2}$ and the x-axis, with $x \geq 0$.

ANSWERS

1. a) $\left(\frac{15}{13}, \frac{105}{52}\right)$ b) $\left(\frac{3}{4}, \frac{3}{2}\right)$ c) $\left(\frac{39}{25}, \frac{206}{75}\right)$ d) $\left(\frac{e^2+1}{2(e^2-1)}, \frac{4(e^3-1)}{9(e^2-1)}\right)$
   e) $\left(0, \frac{3}{2\pi}\right)$ f) $\left(\frac{16}{35}, \frac{16}{35}\right)$ g) $\left(\frac{L}{2}, \frac{16}{9\pi}\right)$
2. a) $\left(\frac{4}{5}, 1\right)$ b) $\left(\frac{21}{13-6\ln 2}, \frac{102}{65-30\ln 2}\right)$ c) $\left(\frac{4(b^3-a^3)}{3\pi(b^2-a^2)}, \frac{4(b^3-a^3)}{3\pi(b^2-a^2)}\right)$ d) $\left(\frac{15\pi+32}{3(\pi+8)}, 0\right)$
   e) $\left(\frac{512\sqrt{2}}{105\pi}, 0\right)$ f) $\left(2, \frac{1}{5}\right)$
3. a) $I_x = \frac{1}{16}e^4 - \frac{1}{16}$, $I_y = \frac{1}{8}e^2 - \frac{1}{8}$, $I_0 = \frac{1}{16}e^4 + \frac{1}{8}e^2 - \frac{3}{16}$
   b) $I_x = \frac{4}{75}$, $I_y = \frac{4}{75}$, $I_0 = \frac{8}{75}$ c) $I_x = \frac{1}{12}$, $I_y = \infty$ and $I_0 = \infty$
4. a) $\frac{-2}{5}$ b) $\frac{2}{a} - \frac{2}{a^2}\ln(1+a)$ c) $\frac{e^{-a^2} - e^{-b^2}}{b^2 - a^2}$ d) No average value (Diverges) e) $\frac{1}{2\pi}$