MATHEMATICS 201-BNK-05
Advanced Calculus
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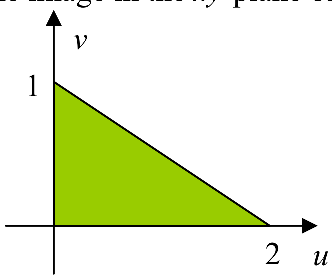
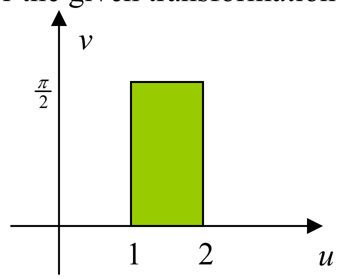
XIX – Change of Variables and Leibnitz's Rule

1. Find the Jacobian of the transformation.
   a) $x = \sin u + \cos v$, $y = -\cos u + \sin v$
   b) $x = u + 2v^2$, $y = 2u^2 - v$
   c) $u = e^x$, $v = ye^{-x}$
   d) $x = uv$, $y = vw$, $z = uw$

2. Sketch the image in the xy-plane of the set R under the given transformation.
   a)
   

   $x = 3u + 4v$
   $y = 4u$

   b)
   

   $x = u\cos v$
   $y = u\sin v$

3. Use the indicated change of variables to evaluate the double integral.
   a) $\iint_R (x - 3y)\,dA$ where R is the triangular region with vertices $(0,0)$, $(2,1)$ and $(1,2)$; $x = 2u + v$, $y = u + 2v$.
   b) $\iint_R x^2\,dA$ where R is the region bounded by the ellipse $9x^2 + 4y^2 = 36$; $x = 2u$, $y = 3v$.
   c) $\iint_R y\sin(xy)\,dA$ where R is the region lying between the graphs of $xy = 1$, $xy = 4$, $y = 1$ and $y = 4$; $x = \frac{u}{v}$, $y = v$.
   d) $\iint_R (x^4 - y^4)e^{xy}\,dA$ where R is the region in the first quadrant enclosed by the hyperbolas $xy = 1$, $xy = 3$, $x^2 - y^2 = 3$, $x^2 - y^2 = 4$; $u = xy$, $v = x^2 - y^2$

4. Use the indicated change of variables to evaluate the triple integral.
   a) $\iiint_S (z-y)^2 xy\,dV$ where S is the solid enclosed by the surfaces $x=1$, $x=3$, $z=y$, $z=y+1$, $xy=2$ and $xy=4$; $u=x$, $v=z-y$, $w=xy$.
   b) $\iiint_S dV$ where S is the solid in the first octant enclosed by the hyperbolic cylinders $xy=1$, $xy=2$, $yz=1$, $yz=3$, $xz=1$, $xz=4$; $u=xy$, $v=yz$, $w=xz$.

5. Use a change of variables to find the volume of the solid region lying below the surface $z=f(x,y)$ and above the plane region R.
   a) $f(x,y)=(x+y)e^{x^2-y^2}$ where R is the rectangle enclosed by the lines $x-y=0$, $x-y=2$, $x+y=0$, $x+y=3$.
   b) $f(x,y)=\sqrt{(x-y)(x+4y)}$ where R is the region bounded by the parallelogram with vertices $(0,0)$, $(1,1)$, $(5,0)$ and $(4,-1)$.
   c) $f(x,y)=\sqrt{x+y}$ where R is the region bounded by the triangle with vertices $(0,0)$, $(a,0)$, $(0,a)$, $a>0$.

6. Show that if R is the triangular region with vertices $(0,0)$, $(1,0)$ and $(0,1)$, then $\iint_R f(x+y)\,dA=\int_0^1 uf(u)\,du$.

7. Use Leibnitz's rule to find the derivative of $F'(x)$. Check your result by evaluating the integral and then differentiating.
   a) $F(x)=\int_0^2 \left(x^2y^3+4xy\right)dy$
   b) $F(x)=\int_{x+1}^{x^2}\left(x^2y-y^2+3\right)dy$
   c) $F(x)=\int_0^x \frac{y-x}{y+x}\,dy$
   d) $F(x)=\int_{\sin x}^{\cos x} xe^{xy}\,dy$

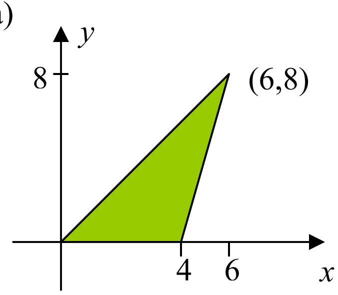
8. Given that $\int_0^b \frac{1}{1+ax}\,dx=\frac{1}{a}\ln(1+ab)$, find a formula for $\int_0^b \frac{x}{(1+ax)^2}\,dx$.

9. Given that $\int \frac{1}{\sqrt{a^2-x^2}}\,dx=\arcsin\left(\frac{x}{a}\right)+C$, find a formula for $\int \frac{1}{\left(a^2-x^2\right)^{\frac{3}{2}}}\,dx$

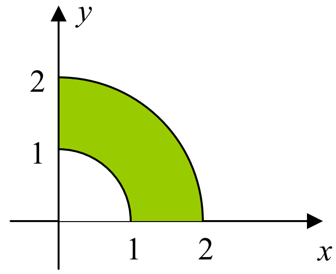
10. Use Leibniz's rule to evaluate the integral $\int_0^\infty \frac{\arctan(ax)}{x\left(1+x^2\right)}\,dx$ where $a\ge 0$

ANSWERS

1. a) $\cos u \cos v + \sin u \sin v$ b) $-1-16uv$ c) 1 d) $2uvw$
2. a)



b)



3. a) -3 b) 6π c) $3\cos 1 - 3\cos 4$ d) $\frac{7}{4}\left(e^3 - e\right)$
4. a) $2\ln 3$ b) $4-4\sqrt{2}-4\sqrt{3}+4\sqrt{6}$
5. a) $\frac{e^6}{4}-\frac{7}{4}$ b) $\frac{100}{9}$ c) $\frac{2}{5}a^{\frac{5}{2}}$
6. Use the change of variables $u = x + y$ and $v = x$.
7. a) $8x+8$ b) $x^5-2x^3-2x^2+7x-2$ c) $1-2\ln 2$
   d) $e^{x\cos x}\cos x - xe^{x\cos x}\sin x - xe^{x\sin x}\cos x - e^{x\sin x}\sin x$
8. $\frac{1}{a^2}\ln\left(1+ab\right)-\frac{b}{a(1+ab)}$
9. $\dfrac{x}{a^2\sqrt{a^2-x^2}}$
10. $\frac{\pi}{2}\ln\left(1+a\right)$