MATHEMATICS 201-BNK-05
Advanced Calculus
Martin Huard
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XIV – Double Integrals in Polar Coordinates

1. Use polar coordinates to evaluate each double integral over the region indicated.
   a) $\iint_R \sqrt{4-x^2-y^2}\,dA$ $\qquad R=\{(x,y)\mid x^2+y^2\le 4, x\ge 0, y\ge 0\}$
   b) $\iint_R \frac{1}{1+x^2+y^2}\,dA$ $\qquad R$ is the sector in the first quadrant bounded by $x^2+y^2=9$, $y=0$ and $y=x$.
   c) $\iint_R (x+y)\,dA$ $\qquad R$ is the region that lies to the left of the y-axis between the circles $x^2+y^2=1$ and $x^2+y^2=4$.

2. Evaluate the double integrals by switching to polar coordinates.
   a) $\int_0^1\int_0^{\sqrt{1-x^2}}\left(x^2+y^2\right)dydx$
   b) $\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2y^2\,dxdy$
   c) $\int_0^4\int_0^{\sqrt{4y-y^2}} x^2\,dxdy$
   d) $\int_0^1\int_y^{\sqrt{y}} \sqrt{x^2+y^2}\,dxdy$
   e) $\int_0^{\frac{a}{\sqrt{2}}}\int_{-\sqrt{a^2-x^2}}^{-x} y\,dydx$

3. Find the area of each region.
   a) One loop of the rose $r=\cos 3\theta$.
   b) The region within both of the circles $r=\cos\theta$ and $r=\sin\theta$.
   c) The region enclosed by the lemniscate $r^2=2a^2\cos 2\theta$

4. Find the volume of each solid.
   a) The solid bounded above and below by the sphere $x^2+y^2+z^2=4$ and bounded laterally by the cylinder $x^2+y^2=1$.
   b) The solid below $z=1-x^2-y^2$, inside of $x^2+y^2-x=0$ and above $z=0$.
   c) The solid bounded by the paraboloid $z=10-3x^2-3y^2$ and the plane $z=4$.
   d) The solid bounded by the cone $z=\sqrt{x^2+y^2}$, the plane $z=0$ and the cylinder $x^2+y^2=25$.
   e) The solid bounded above by the parboloid $z=x^2+y^2$, below by the xy-plane, and inside the cylinder $x^2+y^2-2y=0$.
   f) The solid inside of the surface $r^2+z^2=4$ and outside of the surface $r=2\cos\theta$.

5. Determine the diameter of a hole that is drilled vertically through the center of the solid bounded by the graphs of the equations $z = 25e^{-\frac{x^2+y^2}{4}}$, $z = 0$ and $x^2 + y^2 = 16$ if one-tenth of the volume of the solid is removed.

6. Find a such that the volume inside the hemisphere $z = \sqrt{16 - x^2 - y^2}$ and outside the cylinder $x^2 + y^2 = a^2$ is one-half the volume of the hemisphere.

7. A swimming pool is circular with a 10-m diameter. The depth is constant along east-west lines and increases linearly from 1m at the south end to 3m at the north end. Find the volume of water in the pool.

8. Determine if the following improper integrals converge. If so, to what value?
   a) $\iint\limits_{\mathbb{R}^2} \frac{1}{1+\left(x^2+y^2\right)^2}\,dA$
   b) $\iint\limits_{R} x^2 e^{-x^2-y^2}\,dA$ where R is bounded by $y = x$, $y = -x$ and $x \geq 0$.
   c) $\iint\limits_{R} \frac{1}{\sqrt{x^2+y^2}}\,dA$ where R is the upper half of a disk of radius 2 centered at the origin.
   d) $\iint\limits_{R} \ln\left(x^2+y^2\right)dA$ where R is the unit circle.

9. For what values of k does the integral $\iint\limits_{\mathbb{R}^2} \frac{1}{\left(1+x^2+y^2\right)^k}\,dA$ converge?

ANSWERS

1. a) $\frac{4\pi}{3}$ b) $\frac{\pi \ln 10}{8}$ c) $\frac{-14}{3}$
2. a) $\frac{\pi}{8}$ b) $\frac{4\pi}{3}$ c) 2π d) $\frac{2\sqrt{2}}{45} + \frac{2}{45}$ e) $\frac{-\sqrt{2}}{6}a^3$
3. a) $\frac{\pi}{12}$ b) $\frac{1}{8}\left(\pi - 2\right)$ c) $2a^2$
4. a) $\frac{32}{3}\pi - 4\sqrt{3}\pi$ b) $\frac{5\pi}{32}$ c) 6π d) $\frac{250\pi}{3}$ e) $\frac{3\pi}{2}$ f) $\frac{16}{3}\pi + \frac{64}{9}$
5. $4\sqrt{\ln \frac{10e^4}{9e^4+1}}$ 6. $2\sqrt{4 - 2\sqrt[3]{2}}$ 7. 50π m^3
8. a) $\frac{\pi^2}{2}$ b) $\frac{1}{4} + \frac{\pi}{8}$ c) 2π d) $-\pi$
9. $k > 1$