



MATHEMATICS 201-BNK-05

Advanced Calculus

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X – Extrema for Multivariable Functions

- Find the local maximum and minimum values and saddle points of the function.
 - $f(x, y) = x^2 + xy + y^2 - 9x + 3y - 1$
 - $f(x, y) = x^4 + y^4 + 4xy - 1$
 - $f(x, y) = xy + \frac{8}{x} - \frac{1}{y}$
 - $f(x, y) = x^2y + 4x + xy^2 + 4y$
 - $f(x, y) = \frac{3y}{x^2 + y^2 + 4}$
 - $f(x, y) = xy e^{x+y}$
- Find the absolute maximum and minimum values of f on the set D .
 - $f(x, y) = x^2 + y^2 + xy - 6x - 9y$ $D = \{(x, y) \mid 0 \leq x \leq 6, 0 \leq y \leq 9\}$
 - $f(x, y) = x^4 + y^4 - 4xy + 5$ $D = \{(x, y) \mid |x| \leq 2, |y| \leq 3\}$
 - $f(x, y) = x^2 + y^2 - 6xy + 2$ D is the closed triangular region whose vertices are $(0, 0)$, $(0, 14)$ and $(\frac{14}{3}, 0)$.
 - $f(x, y) = x^2 + 2y^2 - 2x$ $D = \{(x, y) \mid x^2 + y^2 \leq 2\}$
 - $f(x, y) = x^3 - y^3 + 3xy + 2$ D is the closed quadrilateral region whose vertices are $(0, 1)$, $(4, 1)$, $(1, -2)$ and $(-3, -2)$.
- Find the first octant point P on the surface $xyz = 27$ that is closest to the origin. *Hint:* Minimize the distance squared $|PQ|^2$ as a function of x and y .
- Find the point on the plane $3x - 2y + 7z = 5$ that is closest to the point $(5, -5, 6)$.
- Find the dimensions of the box with volume 1000 cm^3 that has minimal surface area.

6. In each of the following problems, the origin is a critical point of f . Determine if $f(\vec{0})$ is a local minimum value, local maximum value, or neither.

- a) $f(x, y, z) = x^2 + 3y^2 + yz + z^2$
- b) $f(x, y, z) = xy + yz + xz - x \sin x$
- c) $f(x, y, z) = (x-1)^3 + 2xy - 3z^2 - 3y^2 e^z - z^2 e^y - 3e^x$
- d) $f(x, y, z, t) = x^2 + y^2 + z^2 + t^2 + xy$

7. Consider the function $f(x, y) = 5x^2 + 6xy + 5y^2$.

- a) Find the critical point.
- b) At the critical point, does the function have a maximum value, minimum value, or is it a saddle point?
- c) Find the two principal curvatures (the eigenvalues of the Hessian matrix) at the critical point.
- d) Find the two eigenvectors associated with the eigenvalues found in (c) (eigenvectors are non-trivial solutions to the matrix equation $H - \lambda I = 0$ for a given eigenvalue λ).
- e) Consider the line, passing through the origin, $\begin{cases} x = t \cos \theta \\ y = t \sin \theta \end{cases}$, where θ is fixed. Find the

curvature of the curve $f(t \cos \theta, t \sin \theta)$ at $t = 0$ using the formula $\kappa = \frac{f''(t)}{(1 + f'(t)^2)^{3/2}}$

from exercise sheet IV question 6.

- f) Find the angle θ which will give the maximal and minimal curvature κ (find the local max and min for $\kappa(\theta)$).
- g) How do the two angles found in (f) compare with the direction of the eigenvectors found in (e)?

ANSWERS

1.
 - a) min of -40 at $(7, -5)$
 - b) local minimum of -3 at $(-1, 1)$ and $(1, -1)$, saddle point at $(0, 0)$
 - c) local maximum of -6 at $(-4, \frac{1}{2})$
 - d) saddle points at $(-2, 2)$ and $(2, -2)$
 - e) local minimum value of $\frac{-3}{4}$ at $(0, -2)$ and local maximum value of $\frac{3}{4}$ at $(0, 2)$
 - f) local maximum value of e^{-2} at $(-1, -1)$ and saddle point at $(0, 0)$
2.
 - a) Abs. max of 54 at $(6, 9)$ and abs. min of -21 at $(1, 4)$
 - b) Abs. max of 126 at $(-2, 3)$ and $(2, -3)$ and abs. min of 3 at $(1, 1)$ and $(-1, -1)$
 - c) Abs. max of 198 at $(0, 14)$ and abs. min of -54 at $(3, 5)$
 - d) Abs. max of 5 at $(-1, 1)$ and $(-1, -1)$ and abs. min of -1 at $(1, 0)$
 - e) Abs max of 77 at $(4, 1)$ and abs. min of 1 at all points on $y = x + 1, -3 \leq x \leq 0$.
3. $(3, 3, 3)$
4. $P(2, -3, -1)$
5. Cube of size 10 cm by 10 cm by 10 cm
6.
 - a) Local min
 - b) Neither
 - c) Local max
 - d) Local min
7.
 - a) $(0, 0)$
 - b) Local minimum
 - c) $\lambda_1 = 4, \lambda_2 = 16$
 - d) $\lambda_1 = 4, \vec{v}_1 = (-1, 1)$ or $(1, -1)$ $\lambda_2 = 16, \vec{v}_2 = (1, 1)$ or $(-1, -1)$
 - e) $\kappa(\theta) = 12 \cos \theta \sin \theta + 10$
 - f) local max $\kappa(\frac{\pi}{4}) = \kappa(\frac{5\pi}{4}) = 16$ and local min $\kappa(\frac{3\pi}{4}) = \kappa(\frac{7\pi}{4}) = 4$
 - g) The angles are the same as for the eigenvectors