MATHEMATICS 201-BNK-05
Advanced Calculus
Martin Huard
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VIII – Differentiability and Chain Rules

1. Show that the function is differentiable at the given point by finding values of ε_1 and ε_2 that satisfy the definition.
   a) $f(x,y) = xy + 3x^2$ at $(2,3)$
   b) $f(x,y) = xy^2$ at (a,b)

2. Find $\frac{dz}{dt}$ or $\frac{dw}{dt}$.
   a) $z = x^2y^3 - x^2 + xy^4$ $\quad x = t^2$ and $y = t^3$
   b) $z = \sqrt{y^2 - x^2}$ $\quad x = \cos t$ and $y = \sin t$
   c) $z = \ln(2x^3 + y^2)$ $\quad x = t^{\frac{1}{3}}$ and $y = \sqrt{t}$ $\quad$ at $t = 5$
   d) $w = x^2y^3z^4$ $\quad x = te^t$, $y = te^{-t}$ and $z = t^2$

3. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.
   a) $z = x^2 + 3xy - 2y^2$ $\quad x = u^2v$ and $y = u^2 + v^2$
   b) $z = e^r \sin\theta$ $\quad r = uv$ and $\theta = \sqrt{u^2 + v^2}$
   c) $z = \arctan\left(\frac{x}{y}\right)$ $\quad x = \sqrt{uv}$ and $y = \frac{1-v}{u}$
   d) $z = \ln(x + \tan y)$ $\quad x = \sin u \cos v$ and $y = \arctan(u + v)$ $\quad$ at $u = \frac{-\pi}{4}$, $v = \frac{\pi}{2}$
   e) $z = 6wxy^3$ $\quad w = uv$, $x = u + 2v$ and $y = u^3v^2$

4. Find $\frac{\partial z}{\partial t}$, $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ where $z = e^{\frac{x}{y}}$, $x = uv + t$, $y = u^2 - v^2t$.

5. Find $\frac{\partial w}{\partial t}$, $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ where $w = 5xyz^2$, $x = 2u + 3v + 4t$, $y = uvt^2$ and $z = u - vt$ when $u = -1$, $v = -2$ and $t = 1$.

6. Find $\frac{\partial w}{\partial \rho}$, $\frac{\partial w}{\partial \theta}$ and $\frac{\partial w}{\partial \phi}$ where $w = x^2 + y^2 - z^2$, $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$.

7. Use the formula seen in class to find $\frac{dy}{dx}$.

   a) $x^2y^3 - 2xy = x^2 + y^3$
   b) $\sin x \cos y = \sin(xy)$
   c) $x^2 + xy + y^2 = \left(x^2 + y^2\right)^2$

8. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

   a) $x^2 + y^2 + z^2 = 3xyz$
   b) $xy = \ln(y+z)$
   c) $x^2 + y^2 = (x+z)^2$

9. For $(x+y)^2 = (x-z)^2$ find $\frac{\partial x}{\partial y}$ and $\frac{\partial x}{\partial z}$.

10. For $x + 2y + 3z = u^2 - v^2$, find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

11. For $x = y^2$ and $x^2 = z$, find $\frac{dy}{dx}$ and $\frac{dz}{dx}$

12. For $x^2 + y^2 - z^2 = 1$ and $x + y + z = 1$, find $\frac{dx}{dy}$ and $\frac{dz}{dy}$.

13. For $z = x^2 - y^2$ and $x^2 + y^2 + z^2 = 1$, find $\frac{dx}{dz}$ and $\frac{dy}{dz}$ at the point $\left(\frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}, 0\right)$

14. For $x^2u + 2v = y^4$ and $2yu^2 - x^2v^2 = 3x$, find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$.

15. For $x = r\cos\theta$ and $y = r\sin\theta$, find $\frac{\partial r}{\partial x}$, $\frac{\partial r}{\partial y}$, $\frac{\partial \theta}{\partial x}$ and $\frac{\partial \theta}{\partial y}$.

16. For $x + y = u + v - w$, $2x - y = 3u + 4v$ and $5y = 5v + 2w$, find $\frac{\partial u}{\partial x}$, $\frac{\partial w}{\partial x}$, $\frac{\partial v}{\partial y}$, $\frac{\partial w}{\partial y}$.

17. For $x = e^r\cos\theta$, $y = e^r\sin\theta$, $z = te^r$, find $\frac{\partial r}{\partial x}$, $\frac{\partial \theta}{\partial x}$, $\frac{\partial t}{\partial y}$, $\frac{\partial t}{\partial z}$.

18. For $x^2 + y^2 = uv$ and $x^2 - y^2 = u + v$, find $\frac{\partial u}{\partial x}$, $\frac{\partial x}{\partial u}$, $\frac{\partial u}{\partial y}$, $\frac{\partial y}{\partial u}$.

19. If f is a differentiable function of two variables and $w = f\left(ay - x, x - ay\right)$, where a is a constant, prove that

$$a\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = 0$$

20. Let $w = f\left(x, y\right)$ where f is a differentiable function. If $x = r\cos\theta$ and $y = r\sin\theta$, show that

a) $\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2$

b) $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r}$

21. Show that a function of the form $z = f\left(x + at\right) + g\left(x - at\right)$ is a solution to the wave equation $\frac{\partial^2 z}{\partial t^2} = a^2\frac{\partial^2 z}{\partial x^2}$. *Hint*: Let $u = x + at$ and $v = x - at$.

22. Suppose that $z = f\left(u\right)$ and $u = g\left(x, y\right)$.

a) Show that $\frac{\partial^2 z}{\partial x^2} = \frac{dz}{du}\frac{\partial^2 u}{\partial x^2} + \frac{d^2 z}{du^2}\left(\frac{\partial u}{\partial x}\right)^2$

b) Find a formula similar to the one in (a) for $\frac{\partial^2 z}{\partial y^2}$.

c) Find a formula similar to the one in (a) for $\frac{\partial^2 z}{\partial x \partial y}$.

23. The height of a right circular cylinder is increasing at the rate of 3 cm/min, and the radius is decreasing at the rate of 2 cm/min. Is the volume increasing or decreasing at the instant the height is 5 cm and the radius is 6 cm?

24. The equation of a perfect gas is $PV = kT$, where V is the volume, T the temperature in degrees Kelvins (K), P the pressure and k a constant. At a certain time, a sample of gas is under a pressure of 50 000 Newtons per square meter (N/m^2), its volume is 0.02 m^3, and its temperature 300 K. If the pressure is increasing at a rate of 1200 N/m^2/min and the volume is decreasing at a rate of 0.001 m^3/min, find the rate at which the temperature is changing.

ANSWERS

1. a) One possible answer is to have $\varepsilon_1 = 3\Delta x$ and $\varepsilon_2 = \Delta x$.
   b) One possible answer is to have $\varepsilon_1 = 2b\Delta y$ and $\varepsilon_2 = a\Delta y + \Delta x\Delta y$.
2. a) $14t^{13} + 13t^{12} - 4t^3$ b) $\frac{2\sin t\cos t}{\sqrt{\sin^2 t - \cos^2 t}}$ c) $\frac{1}{5}$ d) $t^{12}(13-t)e^{-t}$
3. a) $\frac{\partial z}{\partial u} = 2(2x+3y)uv + 2(3x-4y)u$
   $\frac{\partial z}{\partial v} = (2x+3y)u^2 + 2(3x-4y)v$
   b) $\frac{\partial z}{\partial u} = ve^r\sin\theta + \frac{ue^r\cos\theta}{\sqrt{u^2+v^2}}$
   $\frac{\partial z}{\partial v} = ue^r\sin\theta + \frac{ve^r\cos\theta}{\sqrt{u^2+v^2}}$
   c) $\frac{\partial z}{\partial u} = \frac{yu^{\frac{3}{2}}v^{\frac{1}{2}} + 2x(1-v)}{2u^2(x^2+y^2)}$
   $\frac{\partial z}{\partial v} = \frac{yu^{\frac{3}{2}} + 2x\sqrt{v}}{2(x^2+y^2)u\sqrt{v}}$
   d) $\frac{\partial z}{\partial u} = \frac{4}{\pi}$
   $\frac{\partial z}{\partial v} = \frac{2\sqrt{2}+4}{\pi}$
   e) $\frac{\partial z}{\partial u} = 6xy^3v + 6wy^3 + 54wxy^2u^2v^2$
   $\frac{\partial z}{\partial v} = 6xy^3u + 12wy^3 + 36wxy^2u^3v$
4. $\frac{\partial z}{\partial t} = \frac{e^{\frac{y}{x}}(y+xv^2)}{y^2}$ $\frac{\partial z}{\partial u} = \frac{e^{\frac{y}{x}}(yv-2xu)}{y^2}$ $\frac{\partial z}{\partial v} = \frac{e^{\frac{y}{x}}(yu+2xvt)}{y^2}$
5. $\frac{\partial w}{\partial t} = -200$ $\frac{\partial w}{\partial u} = -20$ $\frac{\partial w}{\partial v} = 130$
6. $\frac{\partial w}{\partial \rho} = 2x\sin\phi\cos\theta + 2y\sin\phi\sin\theta - 2z\cos\phi$
   $\frac{\partial w}{\partial \phi} = 2x\rho\cos\phi\cos\theta + 2y\rho\cos\phi\sin\theta + 2z\rho\sin\phi$
   $\frac{\partial w}{\partial \theta} = -2x\rho\sin\phi\sin\theta + 2y\rho\sin\phi\cos\theta$
7. a) $\frac{2x+2y-2xy^3}{3x^2y^2-2x-3y^2}$ b) $\frac{\cos x\cos y - y\cos(xy)}{\sin x\sin y + x\cos(xy)}$ c) $\frac{4x^3+4xy^2-2x-y}{x+2y-4x^2y-4y^3}$
8. a) $\frac{\partial z}{\partial x} = \frac{3yz-2x}{2z-3xy}$, $\frac{\partial z}{\partial y} = \frac{3xz-2y}{2z-3xy}$ b) $\frac{\partial z}{\partial x} = y^2 + yz$, $\frac{\partial z}{\partial y} = xy + xz - 1$
   c) $\frac{\partial z}{\partial x} = \frac{-z}{x+z}$, $\frac{\partial z}{\partial y} = \frac{y}{x+z}$
9. $\frac{\partial x}{\partial y} = \frac{-x-y}{y+z}$, $\frac{\partial x}{\partial z} = \frac{-x+z}{y+z}$
10. $\frac{\partial u}{\partial x} = \frac{1}{2u}$, $\frac{\partial u}{\partial y} = \frac{1}{u}$
11. $\frac{dy}{dx} = \frac{1}{2y}$, $\frac{dz}{dx} = 2x$
12. $\frac{dx}{dy} = \frac{-y-z}{x+z}$, $\frac{dz}{dy} = \frac{y-x}{z+x}$
13. $\frac{\sqrt{2}}{4}$, $\frac{\sqrt{2}}{4}$
14. $\frac{\partial u}{\partial x} = \frac{-2x^3uv+2xv^2+3}{x^4v+4yu}$, $\frac{\partial u}{\partial y} = \frac{4x^2y^3v-2u^2}{x^4v+4yu}$, $\frac{\partial v}{\partial x} = \frac{-2x^3v^2-3x^2-8xyu^2}{2x^4v+8yu}$, $\frac{\partial v}{\partial y} = \frac{x^2u^2+8uy^4}{x^4v+4yu}$
15. $\frac{\partial r}{\partial x} = \cos\theta$, $\frac{\partial r}{\partial y} = \sin\theta$, $\frac{\partial \theta}{\partial x} = \frac{-\sin\theta}{r}$, $\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$
16. $\frac{\partial u}{\partial x} = \frac{6}{13}$. $\frac{\partial w}{\partial x} = \frac{-5}{13}$, $\frac{\partial v}{\partial y} = \frac{23}{13}$, $\frac{\partial w}{\partial y} = \frac{-25}{13}$
17. $\frac{\partial r}{\partial x} = e^{-r}\cos\theta$. $\frac{\partial \theta}{\partial x} = -e^{-r}\sin\theta$, $\frac{\partial t}{\partial y} = -te^{-r}\sin\theta$, $\frac{\partial t}{\partial z} = e^{-r}$
18. $\frac{\partial u}{\partial x} = \frac{2x(u-1)}{u-v}$, $\frac{\partial x}{\partial u} = \frac{v+1}{4x}$, $\frac{\partial u}{\partial y} = \frac{2y(1+u)}{v-u}$, $\frac{\partial y}{\partial u} = \frac{v-1}{4y}$
19. Let $w = f(u,v)$ where $u = ay - x$ and $v = x - ay$.
    Then $\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = -\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}$ and $\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} = a\frac{\partial w}{\partial u} - a\frac{\partial w}{\partial v}$, thus $a\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = 0$

19. b) $\frac{\partial^2 z}{\partial y^2} = \frac{dz}{du}\frac{\partial^2 u}{\partial y^2} + \frac{d^2 z}{du^2}\left(\frac{\partial u}{\partial y}\right)^2$ c) $\frac{\partial^2 z}{\partial x\partial y} = \frac{dz}{du}\frac{\partial^2 u}{\partial y\partial x} + \frac{d^2 z}{du^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$
20. Decreasing at a rate of 12π cm^3/min
21. $\frac{-39}{5} K°$/min