



MATHEMATICS 201-BNK-05

Advanced Calculus

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VII – Partial Derivatives

1. Find $f_x(x, y)$ and $f_y(x, y)$ using the definition of the derivative for the function $f(x, y) = x^2y - y^3 + xy^2$.

2. Find the first partial derivatives of the function.

a) $f(x, y) = x^4y - x^3 + 2x^2y^2$

b) $z = \frac{x^2 + y^2}{x^2 - y^2}$

c) $f(x, y) = e^{x+y} - \ln(xy)$

d) $f(r, \theta) = \frac{r \cos \theta}{1 + r \sin \theta}$

e) $f(x, y) = \arctan\left(\frac{x}{y}\right)$

f) $f(x, y) = \int_y^x e^{-t^2} dt$

g) $f(x, y) = (1+x)^{\cos y}$

h) $f(x, y) = (xy+1)^{\ln x+y}$

i) $f(x, y, z) = z\sqrt{1+x^2+y^2z^2}$

j) $w = \sin^3 x + \cos^3 y + \sec z \tan^4 t$

3. Find the indicated partial derivative.

a) $f(x, y) = \frac{x-xy}{\sqrt{x^2+y^2}}$; $f_x(3, 4)$

b) $f(x, y, z) = xe^{yz}$; $f_z(1, 2, 3)$

4. Given that $w = (ax + by + cz)^n$, where a , b and c are constants, verify that

$$x \frac{\partial w}{\partial x} + y \frac{\partial w}{\partial y} + z \frac{\partial w}{\partial z} = nw$$

5. Given that $z = \frac{e^{x^2+y^2}}{x^2+y^2}$, show that $x \frac{\partial z}{\partial y} - y \frac{\partial z}{\partial x} = 0$.

6. Let $f(x, y) = \sqrt{2x^2 - 7y}$.

a) Find the slope of the surface $z = f(x, y)$ in the x -direction at the point $(4, 1)$.

b) Find the slope of the surface $z = f(x, y)$ in the y -direction at the point $(4, 1)$.

7. Let $f(x, y) = e^{\frac{x}{y}} - e^{\frac{y}{x}}$.

a) Find the slope of the surface $z = f(x, y)$ in the x -direction at the point $(3, 3)$.

b) Find the slope of the surface $z = f(x, y)$ in the y -direction at the point $(3, 3)$.

8. Find the indicated partial derivatives

a) $f(x, y) = x^5y^2 + x^3 - xy^4$; f_{xy} f_{yy}

b) $f(x, y) = x^2e^{y^3}$; f_{yxx} f_{yyx}

c) $z = \sqrt{x^2 - y^2}$; $\frac{\partial^3 z}{\partial x \partial y \partial y}$ $\frac{\partial^3 z}{\partial x^3}$

d) $w = x^a y^b z^c$ $\frac{\partial^5 w}{\partial x^2 \partial y \partial z^2}$

9. Show that the function $z = xe^y + ye^x$ is a solution of the equation

$$\frac{\partial^3 z}{\partial x^3} + \frac{\partial^3 z}{\partial y^3} = x \frac{\partial^3 z}{\partial x \partial y^2} + y \frac{\partial^3 z}{\partial x^2 \partial y}$$

10. The gas law for a fixed mass m of an ideal gas at absolute temperature T , pressure P , and volume V is $PV = nRT$ where R is the gas constant. Show that $\frac{\partial P}{\partial V} \frac{\partial V}{\partial T} \frac{\partial T}{\partial P} = -1$.

11. Show that $w = (a \cos cx + b \sin cx)e^{-kc^2t}$, where a , b , c and k are constants is a solution to the

heat equation $\frac{\partial w}{\partial t} = k \frac{\partial^2 w}{\partial x^2}$.

12. Show that $f(x, y) = \ln(x^2 + y^2)$ satisfies **Laplace's partial differential equation**

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

13. Show that $u = \sin(kx)\sin(akt)$ is a solution to the **wave equation** $u_{tt} = a^2u_{xx}$.

ANSWERS

1. $f_x(x, y) = 2xy + y^2$ $f_y(x, y) = x^2 - 3y^2 + 2xy$
2. a) $f_x(x, y) = 4x^3y - 3x^2 + 4xy^2$ $f_y(x, y) = x^4 + 4x^2y$
 b) $\frac{\partial z}{\partial x} = \frac{-4xy^2}{(x^2 - y^2)^2}$ $\frac{\partial z}{\partial x} = \frac{4x^2y}{(x^2 - y^2)^2}$
- c) $f_x(x, y) = 2xe^{x^2+y} - \frac{1}{x}$ $f_x(x, y) = e^{x^2+y} - \frac{1}{y}$
- d) $f_r(r, \theta) = \frac{\cos \theta}{(1 + r \sin \theta)^2}$ $f_\theta(r, \theta) = \frac{-r \sin \theta - r^2}{(1 + r \sin \theta)^2}$
- e) $f_x(x, y) = \frac{y}{x^2 + y^2}$ $f_y(x, y) = \frac{-x}{x^2 + y^2}$
- f) $f_x(x, y) = e^{-x^2}$ $f_y(x, y) = -e^{-y^2}$
- g) $f_x(x, y) = \cos y (1+x)^{\cos y - 1}$ $f_y(x, y) = -\sin y (1+x)^{\cos y} \ln(1+x)$
- h) $f_x(x, y) = (xy+1)^{\ln xy + y} \left(\frac{1}{x} \ln(xy+1) + \frac{y \ln x + y^2}{xy+1} \right)$ $f_y(x, y) = (xy+1)^{\ln xy + y} \left(\ln(xy+1) + \frac{x \ln x + xy}{xy+1} \right)$
- i) $f_x(x, y, z) = \frac{xz}{\sqrt{1+x^2+y^2z^2}}$ $f_y(x, y, z) = \frac{yz^3}{\sqrt{1+x^2+y^2z^2}}$ $f_z(x, y, z) = \frac{1+x^2+2y^2z^2}{\sqrt{1+x^2+y^2z^2}}$
- j) $\frac{\partial w}{\partial x} = 3 \sin^2 x \cos x$ $\frac{\partial w}{\partial y} = -3 \sin y \cos^2 y$ $\frac{\partial w}{\partial z} = \sec z \tan z \tan^4 t$ $\frac{\partial w}{\partial t} = 4 \sec z \tan^3 t \sec^2 t$
3. a) $\frac{48}{125}$ b) $2e^6$
6. a) $\frac{8}{5}$ b) $\frac{-7}{10}$
7. a) $\frac{2}{3}e$ b) $\frac{-2}{3}e$
8. a) $f_{xy}(x, y) = 10x^4y - 4y^3$ $f_{yy}(x, y) = 2x^5 - 12xy^2$
 b) $f_{yx}(x, y) = 6y^2e^{y^3}$ $f_{yyx}(x, y) = 12xye^{y^3} + 18xy^4e^{y^3}$
 c) $\frac{\partial^3 z}{\partial x \partial y^2} = \frac{x^3 + 2xy^2}{(x^2 - y^2)^{\frac{5}{2}}}$ $\frac{\partial^3 z}{\partial x^3} = \frac{3y^2x}{(x^2 - y^2)^{\frac{5}{2}}}$
 d) $\frac{\partial^5 w}{\partial x^2 \partial y \partial z^2} = a(a-1)bc(c-1)x^{a-2}y^{b-1}z^{c-2}$