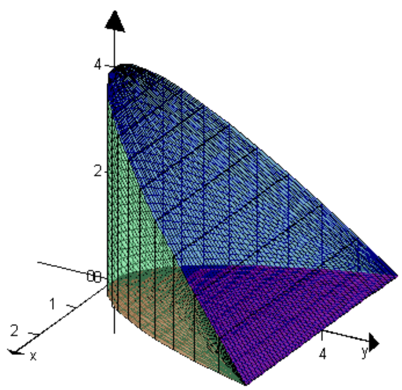
MATHEMATICS 201-BNK-05
Differential Calculus
Martin Huard
May 2, 2010

Test #4
Solutions

*The **only** calculator permitted is the SHARP EL-531.*
*Answer all questions and show **all** your work. Exact answers are required whenever possible.*

Question 1 (7 points)

Rewrite the triple integral $\int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{4-y} f(x,y,z)\,dzdydx$ in 5 different ways.



$$\int_0^4\int_{-\sqrt{y}}^{\sqrt{y}}\int_0^{4-y} f(x,y,z)\,dzdxdy$$
$$\int_0^4\int_{-\sqrt{4-z}}^{\sqrt{4-z}}\int_{x^2}^{4-z} f(x,y,z)\,dydxdz$$
$$\int_{-2}^2\int_0^{4-x^2}\int_{x^2}^{4-z} f(x,y,z)\,dydzdx$$
$$\int_0^4\int_0^{4-z}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dxdydz$$
$$\int_0^4\int_0^{4-y}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dxdzdy$$

Question 2 (8 points)

Find the volume of the solid bounded above by $x^2+y^2+z^2=4z$ and below by $z=1$.

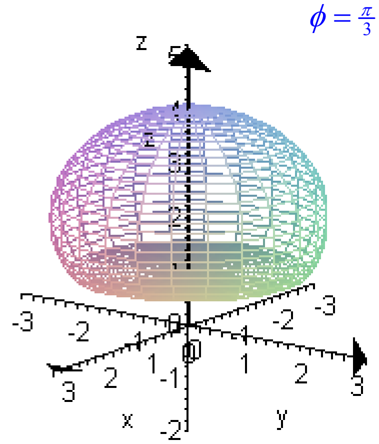
$x^2+y^2+z^2=4z$ $\quad$ $z=1$ $\quad$ Intersection: $4\cos\phi=\sec\phi$

$\rho^2=4\rho\cos\phi$ $\quad$ $\rho\cos\phi=1$ $\quad$ $\cos^2\phi=\frac{1}{4}$

$\rho=4\cos\phi$ $\quad$ $\rho=\sec\phi$ $\quad$ $\cos\phi=\frac{1}{2}$

$\phi=\frac{\pi}{3}$

$$\iiint_S dV=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_{\sec\phi}^{4\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\tfrac{1}{3}\left(64\cos^3\phi-\tfrac{1}{\cos^3\phi}\right)\sin\phi\,d\phi\,d\theta$$
$$=-\int_0^{2\pi}\int_1^{\frac{1}{2}}\tfrac{1}{3}\left(64u^3-\tfrac{1}{u^3}\right)du\,d\theta$$
$$=\int_0^{2\pi}\tfrac{1}{3}\left[16u^4+\tfrac{1}{2u^2}\right]_{\frac{1}{2}}^{1}d\theta$$
$$=9\pi$$



Question 3 (7 points)

Evaluate $\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{x^2+y^2}^{16-x^2-y^2} \sqrt{x^2+y^2}\, dz dy dx$.

$$\begin{aligned}
\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{x^2+y^2}^{16-x^2-y^2} \sqrt{x^2+y^2}\, dz dy dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^3 \int_{r^2}^{16-r^2} r^2\, dz dr d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^3 \left(16r^2 - 2r^4\right) dr d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\tfrac{16}{3} r^3 - \tfrac{2}{5} r^5\right]_0^3 \\
&= \tfrac{234\pi}{5}
\end{aligned}$$

Question 4 (7 points)

Using a change of variables, evaluate $\iint_R \cos(x+y)\cos(2x-y)\, dA$ where R is the parallelogram bounded by the lines $x+y=0$, $x+y=\frac{\pi}{2}$, $2x-y=0$ and $2x-y=\frac{\pi}{4}$.

$$u = x+y \qquad x = \tfrac{u+v}{3}$$

$$v = 2x-y \qquad y = \tfrac{2u-v}{3}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{-1}{3} \end{vmatrix} = -\tfrac{1}{3}$$

$$\begin{aligned}
\iint_R \cos(x+y)\cos(2x-y)\, dA &= \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \cos u \cos v \left|\tfrac{-1}{3}\right| du dv \\
&= \int_0^{\frac{\pi}{4}} \left[\tfrac{1}{3} \sin u \cos v\right]_0^{\frac{\pi}{2}} \\
&= \tfrac{1}{3}\left[\sin v\right]_0^{\frac{\pi}{4}} \\
&= \tfrac{\sqrt{2}}{6}
\end{aligned}$$

Question 5 (4 points)

Find $F'(x)$ using Leibnitz's rule if $F(x) = \int_{\sqrt{x}}^{x^2} e^{x^2+y^2}\, dy$.

$$\begin{aligned}
F'(x) &= \int_{\sqrt{x}}^{x^2} \frac{\partial}{\partial x}\left[e^{x^2+y^2}\right] dy + e^{x^2+\left(x^2\right)^2} \tfrac{d}{dx}\left[x^2\right] - e^{x^2+\left(\sqrt{x}\right)^2} \tfrac{d}{dx}\left[\sqrt{x}\right] \\
&= \int_{\sqrt{x}}^{x^2} 2x e^{x^2+y^2} dy + 2x e^{x^2+x^4} - \tfrac{1}{2\sqrt{x}} e^{x^2+x}
\end{aligned}$$

Question 6 (0 points)

Solve the following differential equation.

$$\left(2xe^{x^2+y^2}-4x^3\right)dx+\left(2ye^{x^2+y^2}+\frac{1}{1+y^2}\right)dy=0$$

$$\frac{\partial M}{\partial y}=4xye^{x^2+y^2} \qquad \frac{\partial N}{\partial x}=4xye^{x^2+y^2}=\frac{\partial M}{\partial y}$$

Thus the differential equation is exact.

$$f(x,y)=\int\left(2xe^{x^2+y^2}-4x^3\right)dx$$

$$=e^{x^2+y^2}-x^4+\phi(y)$$

$$\frac{\partial f}{\partial y}=N$$

$$2ye^{x^2y^2}+\phi'(y)=2ye^{x^2y^2}+\frac{1}{1+y^2}$$

$$\phi(y)=\arctan y$$

Thus the solution is $e^{x^2+y^2}-x^4+\arctan y=K$

Question 7 (8 points)

Solve the following differential equation.

$$\frac{dy}{dx}+\frac{y}{x}=xy^3$$

This is a Bernouilli differential equation.

Let $z=\frac{1}{y^2}$, $\frac{dz}{dx}=\frac{-2}{y^3}\frac{dy}{dx}$

$$\frac{-2}{y^3}\frac{dy}{dx}-\frac{2}{xy^2}=-2x$$

$$\frac{dz}{dx}-\frac{2z}{x}=-2x \qquad \text{which is linear}$$

$$\mu=e^{\int\frac{-2}{x}dx}=e^{-2\ln x}=\frac{1}{x^2}$$

$$\frac{z}{x^2}=\int\frac{-2}{x}dx$$

$$\frac{1}{x^2y^2}=-2\ln|x|+C$$

$$y^2=\frac{1}{x^2\left(C-2\ln|x|\right)}$$

Question 8 (8 points)
Solve the following differential equation.

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 10 + \sin x$$

characteristic equation: $m^3 - 4m^2 + 5m = 0$

$$m\left(m^2 - 4m + 5\right) = 0$$

$$m = 0, \frac{4 \pm \sqrt{16-20}}{2}$$

$$= 0, 2 \pm i$$

$$y_h = K_1 + e^{2x}\left(K_2 \cos x + K_3 \sin x\right)$$

$$y_p = Ax + B\sin x + C\cos x$$

$$y_p' = A + B\cos x - C\sin x$$

$$y_p'' = -B\sin x - C\cos x$$

$$y_p''' = -B\cos x + C\sin x$$

$$y''' - 4y'' + 5y' = \sin x$$

$$-B\cos x + C\sin x + 4B\sin x + 4C\cos x + 5A + 5B\cos x - 5C\sin x = 10 + \sin x$$

$$5A + \left(4B - 4C\right)\sin x + \left(4B + 4C\right)\cos x = 10 + \sin x$$

Hence $5A = 10$ $\quad A = 2$

$4B + 4C = 0$ $\quad B = \frac{1}{8},\ C = \frac{-1}{8}$

$4B - 4C = 1$

Thus the solution is

$$y = K_1 + e^{2x}\left(K_2 \cos x + K_3 \sin x\right) + 2x + \tfrac{1}{8}\sin x - \tfrac{1}{8}\cos x$$