MATHEMATICS 201-BNK-05
Advanced Calculus
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Semester Review

SOLUTIONS and HINTS

1. Use the definition of a limit to prove the following statements.

a) $\lim_{x\to 2}\left(3x^2-4x+1\right)=5$

Let $\varepsilon>0$ be given.

$$\left|f(x)-L\right|=\left|3x^2-4x+1-5\right| = |x-2||3x+2| < 11\delta=\varepsilon$$

Hyp: $0<|x-2|<\delta$

Let $\delta\le 1$, then $-1<x-2<1$

$1<x<3$

$5<3x+2<11$

If $\delta=\min\left\{1,\frac{\varepsilon}{11}\right\}$ then $0<|x-2|<\delta\Rightarrow\left|f(x)-5\right|<\varepsilon$, ergo $\lim_{x\to 2}\left(3x^2-4x+1\right)=5$

b) $\lim_{x\to 4}\left(\sqrt{x}+x\right)=6$

Let $\varepsilon>0$ be given.

$$\left|f(x)-L\right|=\left|\sqrt{x}+x-6\right| = \left|\left(\sqrt{x}-2\right)\tfrac{\sqrt{x}+2}{\sqrt{x}+2}\right|+|x-4| = \tfrac{|x-4|}{\sqrt{x}+2}+|x-4| < \tfrac{\delta}{\sqrt{3}+2}+\delta$$

Hyp: $0<|x-4|<\delta$

Let $\delta\le 1$, then

$-1<x-4<1$

$3<x<5$

$\sqrt{3}+2<\sqrt{x}+2<\sqrt{5}+2$

If $\delta=\min\left\{1,\frac{\varepsilon\left(\sqrt{3}+2\right)}{\sqrt{3}+3}\right\}$ then $0<|x-4|<\delta\Rightarrow\left|f(x)-6\right|<\varepsilon$, ergo $\lim_{x\to 4}\left(\sqrt{x}+x\right)=6$.

c) $\lim_{x\to 3^-}\sqrt{9-x^2}=0$

Let $\varepsilon>0$ be given.

$$\left|f(x)-L\right|=\left|\sqrt{9-x^2}\right| = \sqrt{3-x}\sqrt{3+x} < \sqrt{\delta}\sqrt{6}=\varepsilon$$

Hyp: $-\delta<x-3<0$

$0<3-x<\delta$

Let $\delta\le 1$, then

$-1<x-3<0$

$5<x+3<6$

$\sqrt{5}<\sqrt{x+3}<\sqrt{6}$

If $\delta=\min\left\{1,\frac{\varepsilon^2}{6}\right\}$ then $-\delta<x-3<0\Rightarrow\left|f(x)-0\right|<\varepsilon$, ergo $\lim_{x\to 3^-}\sqrt{9-x^2}=0$.

d) $\lim_{x \to 2^+} \frac{1-x}{x-2} = -\infty$

Let $N < 0$ be given.

$f(x) = \frac{1-x}{x-2}$

$< \frac{-1}{\delta} = N$

Hyp: $0 < x - 2 < \delta$

$\frac{1}{\delta} < \frac{1}{x-2}$

Let $\delta \le 1$, then

$0 < x - 2 < 1$

$2 < x < 3$

$-2 < 1 - x < -1$

Thus $\frac{1-x}{x-2} < \frac{1-x}{\delta} < \frac{-1}{\delta}$

If $\delta = \min\{1, \frac{-1}{N}\}$ then $0 < x - 2 < \delta \Rightarrow f(x) < N$, ergo $\lim_{x \to 4}\left(\sqrt{x} + x\right) = 6$.

e) $\lim_{x \to -\infty} \frac{x^2}{x^2+1} = 1$

Let $\varepsilon > 0$ be given.

$|f(x) - L| = \left|\frac{x^2}{x^2+1} - 1\right|$

$= \frac{1}{x^2+1}$

$< \frac{1}{M^2+1} = \varepsilon$

Hyp: $x < M < 0$

$0 < -M < -x$

$1 < M^2 + 1 < x^2 + 1$

$\frac{1}{x^2+1} < \frac{1}{M^2+1}$

If $M = \min\left\{0, \sqrt{\frac{1}{\varepsilon} - 1}\right\}$ then $x < M \Rightarrow |f(x) - 1| < \varepsilon$, ergo $\lim_{x \to -\infty} \frac{x^2}{x^2+1} = 1$.

f) $\lim_{x \to \infty}\left(x^3 + x\right) = \infty$

Let $N > 0$ be given.

$f(x) = x^3 + x$

$> M^3 + M$

$> 1 + M = N$

Hyp: $x > M > 0$

Let $M > 1$, then $M^3 > 1$

If $M = \max\{1, N - 1\}$ then $x > M \Rightarrow f(x) > N$, ergo $\lim_{x \to \infty}\left(x^3 + x\right) = \infty$.

g) $\lim_{x \to -3} \frac{x}{x^2+6x+9} = -\infty$

Let $N < 0$ be given.

$f(x) = \frac{x}{x^2+6x+9}$

$= \frac{x}{(x+3)^2}$

$< \frac{-2}{\delta^2}$

Hyp: $0 < |x+3| < \delta$

$\frac{1}{\delta^2} < \frac{1}{(x+3)^2}$

Let $\delta \le 1$, then

$-1 < x + 3 < 1$

$-4 < x < -2$

Thus $\frac{x}{(x+3)^2} < \frac{x}{\delta^2} < \frac{-2}{\delta}$

If $\delta = \min\left\{1, \sqrt{\frac{-2}{N}}\right\}$ then $0 < |x-3| < \delta \Rightarrow f(x) < N$, ergo $\lim_{x \to -3} \frac{x}{x^2+6x+9} = -\infty$.

h) $\lim_{x\to a}\left(x^2-5x\right)=a^2-5a$

Let $\varepsilon>0$ be given.

$$\begin{aligned}\left|f(x)-L\right| &= \left|x^2-5x-\left(a^2-5a\right)\right| \\ &= \left|(x-a)(x+a)-5(x-a)\right| \\ &= |x-a||x+a-5| \\ &< \delta\left(2|a|+6\right)\end{aligned}$$

Hyp: $0<|x-a|<\delta$

Let $\delta\le 1$, then

$$\begin{aligned} -1 &< x-a < 1 \\ 2a-6 &< x+a-5 < 2a-4 \\ -2|a|-6 &< x+a-5 < 2|a|+4 \\ |x+a-5| &< 2|a|+6\end{aligned}$$

If $\delta=\min\left\{1,\frac{\varepsilon}{2|a|+6}\right\}$ then $0<|x-a|<\delta \Rightarrow \left|f(x)-\left(a^2-5a\right)\right|<\varepsilon$,

ergo $\lim_{x\to a}\left(x^2-5x\right)=a^2-5a$.

2. Use the definition of a limit to show that the following statements are false.

a) $\lim_{x\to 3}\left(x^2-x+1\right)=2$

Let $\varepsilon=1$. Then for all values of δ, there will be values of x on the interval $(3-\delta,3+\delta)$ such that $x>3$. For these values,

$$\begin{aligned} x &> 3 \\ x-1 &> 2 \\ x^2-x &> 6 \\ x^2-x-1 &> 5\end{aligned}$$

thus $\left|f(x)-2\right|=\left|x^2-x-1\right|>5>\varepsilon=1$. Hence $\lim_{x\to 3}\left(x^2-x+1\right)\ne 2$

b) $\lim_{x\to -1}\frac{x}{x+1}=\infty$

Let $N=10$. Then for all values of δ, there will be values of x on the interval $(-1-\delta,-1+\delta)$ such that $-1<x<\frac{-1}{2}$. For these values,

$$\begin{aligned} 0 &< x+1 < \tfrac{1}{2} \\ 2 &< \tfrac{1}{x+1} \\ \tfrac{x}{x+1} &< 2x < -1\end{aligned}$$

thus $f(x)<-1<N=10$. Hence $\lim_{x\to -1}\frac{x}{x+1}\ne\infty$

c) $\lim_{x\to\infty}\frac{x^2}{x+1}=1$

Let $\varepsilon=1$. Then for all values of *M*, there will be values of *x* on the interval (M,∞) such that $x>5$. For these values,

$$\tfrac{1}{x}+\tfrac{1}{x^2}<\tfrac{1}{5}+\tfrac{1}{25}$$
$$\tfrac{x+1}{x^2}<\tfrac{6}{25}$$
$$\tfrac{x^2}{x+1}>\tfrac{25}{6}$$
$$\tfrac{x^2}{x+1}-1>\tfrac{19}{6}$$

thus $\left|f(x)-1\right|=\left|\tfrac{x^2}{x+1}-1\right|>\tfrac{19}{6}>\varepsilon=1$. Hence $\lim_{x\to\infty}\frac{x^2}{x+1}\neq 1$

d) $\lim_{x\to\infty}\left(x-x^2\right)=\infty$

Let $N=10$. Then for all values of *M*, there will be values of *x* on the interval (M,∞) such that $x>1$. For these values,

$$x^2>x$$
$$x^2-x>0$$
$$x-x^2<0$$

thus $f(x)=x-x^2<0<N=10$. Hence $\lim_{x\to\infty}\left(x-x^2\right)\neq\infty$

e) $\lim_{x\to 2^-}f(x)=5$ where $f(x)=\begin{cases}x^2 & x<2\\ 7-x & x\geq 2\end{cases}$

Let $\varepsilon=1$. Then for all values of δ, there will be values of *x* on the interval $(2-\delta,2)$ such that $0<x<2$. For these values,

$$0<x<2$$
$$0<x^2<4$$
$$-5<x^2-5<-1$$

thus $\left|f(x)-5\right|=\left|x^2-5\right|>1=\varepsilon$. Hence $\lim_{x\to 2^-}f(x)\neq 5$

3. Use the formal definition for continuity to prove the continuity of *f* at the given point.

a) $f(x)=2x^2+3x-4$ at $x=-1$

Let $\varepsilon>0$ be given.

$$\begin{aligned}\left|f(x)-f(-1)\right|&=\left|2x^2+3x-4+5\right|\\&=\left|x+1\right|\left|2x+1\right|\\&<2\delta=\varepsilon\end{aligned}$$

Hyp: $\left|x+1\right|<\delta$

Let $\delta\leq\tfrac{1}{2}$, then

$$\begin{aligned}-\tfrac{1}{2}<x+1&<\tfrac{1}{2}\\-\tfrac{3}{2}<x&<-\tfrac{1}{2}\\-2<2x+1&<0\\0<\left|2x+1\right|&<2\end{aligned}$$

If $\delta=\min\left\{\tfrac{1}{2},\tfrac{\varepsilon}{2}\right\}$ then $\left|x+1\right|<\delta\Rightarrow\left|f(x)-f(-1)\right|<\varepsilon$, ergo *f* is continuous.

b) $f(x)=\dfrac{12}{x^2}$ at $x=2$

Let $\varepsilon>0$ be given.

$$\begin{aligned}|f(x)-f(2)| &= \left|\tfrac{12}{x^2}-3\right| \\ &= \tfrac{3}{x}|x-2||x+2| \\ &< 9\delta=\varepsilon\end{aligned}$$

Hyp: $|x-2|<\delta$

Let $\delta\le 1$, then $-1<x-2<1$

$$1<x<3$$
$$\tfrac{1}{3}<\tfrac{1}{x}<1$$
$$3<x+2<5$$

If $\delta=\min\{1,\frac{\varepsilon}{9}\}$ then $|x-2|<\delta\Rightarrow|f(x)-f(2)|<\varepsilon$, ergo f is continuous.

4. Show that the following functions are discontinuous at the given point.

a) $f(x)=\begin{cases}2x+1 & \text{if } x<-2\\ x-2 & \text{if } x\ge -2\end{cases}$ at $x=-2$

Let $\varepsilon=\frac{1}{2}$. Then for all values of δ, there will be values of x on the interval $(-2-\delta,-2+\delta)$ such that $\frac{-5}{2}<x<-2$. For these values,

$$\tfrac{-9}{4}<x<-2$$
$$\tfrac{1}{2}<2x+1<1$$

thus $|f(x)-f(2)|=|2x+5|>\frac{1}{2}=\varepsilon$. Hence f is discontinuous at $x=-2$

b) $f(x)=\begin{cases}\frac{x-3}{x} & \text{if } x\le 3\\ x & \text{if } x>3\end{cases}$ at $x=3$

Let $\varepsilon=1$. Then for all values of δ, there will be values of x on the interval $(3-\delta,3+\delta)$ such that $x>3$.

For these values, $|f(x)-f(3)|=|x|>3>\varepsilon=1$. Hence f is discontinuous at $x=3$.

5. Show that the functions $f(x)=\dfrac{1}{\sqrt{x}}$ is continuous everywhere for $x>0$.

Let $a>0$ be any value and let $\varepsilon>0$ be given.

$$\begin{aligned}|f(x)-f(a)| &= \left|\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{a}}\right| \\ &= \left|\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x}\sqrt{a}}\cdot\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}\right| \\ &= \frac{|x-a|}{\sqrt{x}\sqrt{a}\left|\sqrt{x}+\sqrt{a}\right|} \\ &< \delta\frac{\sqrt{2}}{\sqrt{a}}\frac{1}{\sqrt{a}}\frac{\sqrt{2}}{(1+\sqrt{2})\sqrt{a}}=\frac{2\delta}{(1+\sqrt{2})a^{\frac{3}{2}}}=\varepsilon\end{aligned}$$

Hyp: $|x-a|<\delta$

Let $\delta\le\frac{a}{2}$, then

$$-\tfrac{a}{2}<x-a<\tfrac{a}{2}$$
$$\tfrac{a}{2}<x<\tfrac{3a}{2}$$
$$\sqrt{\tfrac{2}{3a}}<\tfrac{1}{\sqrt{x}}<\sqrt{\tfrac{2}{a}}$$
$$\tfrac{\sqrt{a}}{\sqrt{2}}+\sqrt{a}<\sqrt{x}+\sqrt{a}<\tfrac{\sqrt{3a}}{\sqrt{2}}+\sqrt{a}$$
$$\tfrac{\sqrt{2}}{(\sqrt{3}+\sqrt{2})\sqrt{a}}<\tfrac{1}{\sqrt{x}+\sqrt{a}}<\tfrac{\sqrt{2}}{(1+\sqrt{2})\sqrt{a}}$$

If $\delta = \min\left\{\frac{a}{2}, \frac{(1+\sqrt{2})a^{\frac{3}{2}}\varepsilon}{2}\right\}$ then $|x-a| < \delta \Rightarrow \left|\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{a}}\right| < \varepsilon$, ergo *f* is continuous everwhere.

6. Consider the curve given by $\vec{r}(t) = \left(t, \frac{1}{t}, \sqrt{t}\right)$.

   a) Evaluate $\lim_{t\to 9} \vec{r}(t)$.

   $$\lim_{t\to 9}\vec{r}(t) = \lim_{t\to 9}\left(t,\tfrac{1}{t},\sqrt{t}\right) = \left(\lim_{t\to 9} t, \lim_{t\to 9}\tfrac{1}{t}, \lim_{t\to 9}\sqrt{t}\right) = \left(9,\tfrac{1}{9},3\right)$$

   b) Evaluate $\int \vec{r}(t)\,dt$.

   $$\int \vec{r}(t)\,dt = \int\left(t,\tfrac{1}{t},\sqrt{t}\right)dt = \left(\int t\,dt, \int \tfrac{1}{t}\,dt, \int\sqrt{t}\,dt\right) = \left(\tfrac{1}{2}t^2, \ln|t|, \tfrac{2}{3}t^{\frac{3}{2}}\right)$$

   c) Find the unit tangent, normal and binormal vectors at $t=1$.

   $$\vec{r}'(t) = \left(1,\tfrac{-1}{t^2},\tfrac{1}{2\sqrt{t}}\right) \qquad \left\|\vec{r}'(t)\right\| = \left\|\left(1,\tfrac{-1}{t^2},\tfrac{1}{2\sqrt{t}}\right)\right\| = \sqrt{1+\tfrac{1}{t^4}+\tfrac{1}{4t}} = \tfrac{\sqrt{4t^4+t^3+4}}{2t^2}$$

   $$\vec{T}(t) = \frac{\vec{r}'(t)}{\left\|r'(t)\right\|} = \frac{2t^2}{\sqrt{4t^4+t^3+4}}\left(1,\frac{-1}{t^2},\frac{1}{2\sqrt{t}}\right)$$

   $$\vec{T}(1) = \frac{2}{3}\left(1,-1,\frac{1}{2}\right) = \left(\frac{2}{3},\frac{-2}{3},\frac{1}{3}\right)$$

   $$\vec{T}'(t) = \frac{4t\sqrt{4t^4+t^3+4} - 2t^2\,\frac{16t^3+3t^2}{2\sqrt{4t^4+t^3+4}}}{\left(4t^4+t^3+4\right)}\left(1,\frac{-1}{t^2},\frac{1}{2\sqrt{t}}\right) + \frac{2t^2}{\sqrt{4t^4+t^3+4}}\left(0,\frac{2}{t^3},\frac{1}{4t^{\frac{3}{2}}}\right)$$

   $$\vec{T}'(1) = \frac{12-\frac{19}{3}}{9}\left(1,-1,\frac{1}{2}\right) + \frac{2}{3}\left(0,2,\frac{1}{4}\right) = \left(\frac{17}{27},\frac{19}{27},\frac{4}{27}\right)$$

   $$\vec{N}(1) = \frac{\vec{T}'(1)}{\left\|\vec{T}'(1)\right\|} = \frac{\sqrt{74}}{222}(17,19,4) = \left(\frac{17\sqrt{74}}{222},\frac{19\sqrt{74}}{222},\frac{2\sqrt{74}}{111}\right)$$

   $$\vec{B}(1) = \vec{T}(1)\times\vec{N}(1) = \frac{1}{3}\frac{\sqrt{74}}{222}\begin{vmatrix} i & j & k \\ 2 & -2 & 1 \\ 17 & 19 & 4 \end{vmatrix} = \frac{\sqrt{74}}{74}(-3,1,8)$$

   d) Find the equation for the tangent line, the normal plane and the osculating plane, at $t=1$.

   Tangent line: $l_T : \frac{x-1}{2} = \frac{1-y}{2} = z-1$

   Normal plane: $\pi_N : 2x-2y+z=1$

   Osculating plane $\pi_O : 3x-y-8z=-6$

7. Consider the curve given by $\vec{r}(t)=\left(e^t-t,\sqrt{7}e^{\frac{1}{2}t},3e^{\frac{1}{2}t}\right)$.

   a) Find the arc length of the graph of $\vec{r}(t)$ from $t=0$ to $t=1$.

$$s=\int_0^1\left\|r'(t)\right\|dt=\int_0^1\left\|\left(e^t-1,\tfrac{\sqrt{7}}{2}e^{\frac{1}{2}t},\tfrac{3}{2}e^{\frac{1}{2}t}\right)\right\|dt$$
$$=\int_0^1\sqrt{\left(e^t-1\right)^2+\tfrac{7}{4}e^t+\tfrac{9}{4}e^t}\,dt$$
$$=\int_0^1\sqrt{e^{2t}+2e^t+1}\,dt$$
$$=\int_0^1\left(e^t+1\right)dt$$
$$=\left[e^t+t\right]_0^1$$
$$=e$$

   b) Find the curvature at $t=0$.

$$\vec{r}'(t)=\left(e^t-1,\tfrac{\sqrt{7}}{2}e^{\frac{1}{2}t},\tfrac{3}{2}e^{\frac{1}{2}t}\right)\qquad \vec{r}'(0)=\left(0,\tfrac{\sqrt{7}}{2},\tfrac{3}{2}\right)$$
$$\vec{r}''(t)=\left(e^t,\tfrac{\sqrt{7}}{4}e^{\frac{1}{2}t},\tfrac{3}{4}e^{\frac{1}{2}t}\right)\qquad \vec{r}''(0)=\left(1,\tfrac{\sqrt{7}}{4},\tfrac{3}{4}\right)$$
$$\kappa(0)=\frac{\left\|\vec{r}'(0)\times\vec{r}''(0)\right\|}{\left\|\vec{r}'(0)\right\|^3}=\frac{\left\|\left(0,\tfrac{\sqrt{7}}{2},\tfrac{3}{2}\right)\times\left(1,\tfrac{\sqrt{7}}{4},\tfrac{3}{4}\right)\right\|}{\left\|\left(0,\tfrac{\sqrt{7}}{2},\tfrac{3}{2}\right)\right\|^3}$$
$$=\frac{\left\|\left(0,\tfrac{3}{2},\tfrac{-\sqrt{7}}{2}\right)\right\|}{8}=\frac{1}{4}$$

   c) Find the unit tangent, normal and binormal vectors at $t=0$.

$$\vec{T}(t)=\frac{\vec{r}'(t)}{\left\|r'(t)\right\|}=\frac{1}{e^t+1}\left(e^t-1,\tfrac{\sqrt{7}}{2}e^{\frac{1}{2}t},\tfrac{3}{2}e^{\frac{1}{2}t}\right)$$
$$\vec{T}(0)=\frac{1}{2}\left(0,\tfrac{\sqrt{7}}{2},\tfrac{3}{2}\right)=\left(0,\tfrac{\sqrt{7}}{4},\tfrac{3}{4}\right)$$
$$\vec{T}'(t)=\frac{-e^t}{\left(e^t+1\right)^2}\left(e^t-1,\tfrac{\sqrt{7}}{2}e^{\frac{1}{2}t},\tfrac{3}{2}e^{\frac{1}{2}t}\right)+\frac{1}{e^t+1}\left(e^t,\tfrac{\sqrt{7}}{4}e^{\frac{1}{2}t},\tfrac{3}{4}e^{\frac{1}{2}t}\right)$$
$$\vec{T}'(0)=\frac{-1}{4}\left(0,\tfrac{\sqrt{7}}{2},\tfrac{3}{2}\right)+\frac{1}{2}\left(1,\tfrac{\sqrt{7}}{4},\tfrac{3}{4}\right)=\left(\tfrac{1}{2},0,0\right)$$
$$\vec{N}(0)=\frac{\vec{T}'(0)}{\left\|\vec{T}'(0)\right\|}=\frac{1}{\frac{1}{2}}\left(\tfrac{1}{2},0,0\right)=(1,0,0)$$
$$\vec{B}(0)=\vec{T}(0)\times\vec{N}(0)=\begin{vmatrix}i & j & k\\ 0 & \frac{\sqrt{7}}{4} & \frac{3}{4}\\ 1 & 0 & 0\end{vmatrix}=\left(0,\tfrac{3}{4},\tfrac{-\sqrt{7}}{4}\right)$$

d) Find the equation for the tangent line, the normal plane and the osculating plane, at the given point.

Tangent line: $l_T : (x,y,z) = (1,\sqrt{7},3) + t(0,\sqrt{7},3)$

Normal plane: $\pi_N : \sqrt{7}y + 3z = 16$

Osculating plane $\pi_O : 3y - \sqrt{7}z = 0$

8. Reparametrize the curve $\vec{r}(t) = (t^3, t^2+1)$ with respect to arc length measured from the point where $t = 0$ in the direction of increasing *t*.

$$s = \int_0^t \|r'(u)\| du = \int_0^t \|(3u^2, 2u)\| du$$
$$= \int_0^t \sqrt{9u^4 + 4u^2}\, du$$
$$= \int_0^t u\sqrt{9u^2+4}\, du$$
$$= \tfrac{1}{27}(9u^2+4)^{\frac{3}{2}}\Big]_0^t$$
$$= \tfrac{1}{27}(9t^2+4)^{\frac{3}{2}} - \tfrac{8}{27}$$

$$27s + 8 = (9t^2+4)^{\frac{3}{2}}$$
$$t = \tfrac{1}{3}\sqrt{(27s+8)^{\frac{2}{3}} - 4}$$

Thus $\vec{r}(s) = \left(\tfrac{1}{27}\left((27s+8)^{\frac{2}{3}} - 4\right)^{\frac{3}{2}}, \tfrac{1}{9}(27s+8)^{\frac{2}{3}} + \tfrac{5}{9} \right)$

9. Find the equation of the osculating circle for the given curve at the given point.

a) $y = \cos x$ at $(0,1)$.

$\vec{r}(t) = (t, \cos t, 0)$

$$\kappa(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} = \frac{\|(1,-\sin t,0)\times(0,-\cos t,0)\|}{\|(1,-\sin t,0)\|^3}$$
$$= \frac{\|(0,0,-\cos t)\|}{(1+\sin^2 t)^{\frac{3}{2}}} = \frac{|\cos t|}{(1+\sin^2 t)^{\frac{3}{2}}}$$

$$\kappa(0) = 1$$

Ergo, the osculating circle is a circle of radius 1 centered on $(0,0)$, $x^2 + y^2 = 1$.

b) $x^2 - y^2 = 4$ at $(2,0)$

$\vec{r}(t) = \left(\sqrt{t^2+4}, t, 0\right)$

$$\kappa(t)=\frac{\left\|\vec{r}'(t)\times\vec{r}''(t)\right\|}{\left\|\vec{r}'(t)\right\|^3}=\frac{\left\|\left(\frac{t}{\sqrt{t^2+4}},1,0\right)\times\left(\frac{4}{\left(t^2+4\right)^{\frac{3}{2}}},0,0\right)\right\|}{\left\|\left(\frac{t}{\sqrt{t^2+4}},1,0\right)\right\|^3}$$

$$=\frac{\left\|\left(0,0,\frac{-4}{\left(t^2+4\right)^{\frac{3}{2}}}\right)\right\|}{\left(\frac{2t^2+4}{t^2+4}\right)^{\frac{3}{2}}}=\frac{\sqrt{2}}{\left(t^2+2\right)^{\frac{3}{2}}}$$

$\kappa(0)=\frac{1}{2}$

Ergo, the osculating circle is a circle of radius 2 centered on $(2,0)$, $(x-4)^2+y^2=4$.

12. Find the limit, and, if it exists prove it using the definition, and, if not, show that it does not exist.

a) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{5xy^3}{x^2+3y^2}$

Let $\varepsilon>0$ be given.

$$\left|f(x,y)-0\right|=\left|\frac{5xy^3}{x^2+3y^2}\right|=5|x||y|\frac{y^2}{x^2+3y^2}<5\delta^2$$

Hyp: $0<\sqrt{x^2+y^2}<\delta$

$x^2\le x^2+y^2<\delta^2$, so $|x|<\delta$

$y^2\le x^2+y^2<\delta^2$, so $|y|<\delta$

$y^2\le x^2+y^2\le x^2+3y^2$, $\frac{y^2}{x^2+3y^2}\le 1$

If $\delta=\sqrt{\frac{\varepsilon}{5}}$ then $0<\sqrt{x^2+y^2}<\delta\Rightarrow\left|f(x,y)-0\right|<\varepsilon$, ergo $\displaystyle\lim_{(x,y)\to(0,0)}\frac{5xy^3}{x^2+3y^2}=0$.

b) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x+y^2}{x^2+y^2}$

Let $f(x,y)=\frac{x+y^2}{x^2+y^2}$

Along the x-axis, $f(x,0)=\frac{x}{x^2}=\frac{1}{x}$,

$\displaystyle\lim_{x\to0^+}f(x,0)=\lim_{x\to0^+}\tfrac{1}{x}=\infty$ case $\frac{b}{0}$

$\displaystyle\lim_{x\to0^-}f(x,0)=\lim_{x\to0^-}\tfrac{1}{x}=-\infty$ case $\frac{b}{0}$

Hence $\displaystyle\lim_{x\to0}f(x,0)\nexists$, ergo, $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x+y^2}{x^2+y^2}\nexists$

c) $\displaystyle\lim_{(x,y,z)\to(2,1,-1)}(x+2y-5z)$

Let $\varepsilon>0$ be given.

$$\begin{aligned}|f(x,y,z)-9| &= |x+2y-5z-9| \\ &= |x-2+y-1-5(z+1)| \\ &\le |x-2|+|y-1|+5|z+1| \\ &< 7\delta\end{aligned}$$

Hyp: $0<\sqrt{(x-2)^2+(y-1)^2+(z+1)^2}<\delta$

Thus $|x-2|<\delta$

$|y-1|<\delta$

$|z+1|<\delta$

If $\delta=\frac{\varepsilon}{7}$ then $0<\sqrt{(x-2)^2+(y-1)^2+(z+1)^2}<\delta \Rightarrow |f(x,y,z)-9|<\varepsilon$,

ergo $\lim\limits_{(x,y,z)\to(2,1,-1)}(x+2y-5z)=9$.

d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y-xy^2}{x^4+y^4}$

Let $f(x,y)=\dfrac{x^2y-xy^2}{x^4+y^4}$

Along the x-axis, $f(x,0)=\dfrac{0}{x^4}=0$, $\lim\limits_{x\to0} f(x,0)=\lim\limits_{x\to0} 0=0$

Along the y-axis, $f(0,y)=\dfrac{0}{y^4}=0$, $\lim\limits_{y\to0} f(0,y)=\lim\limits_{y\to0} 0=0$

Along the line $y=mx$, $f(x,mx)=\dfrac{mx^3-m^2x^3}{x^4+mx^4}=\dfrac{m-m^2}{x(m+1)}$,

$$\lim_{x\to0^+} f(x,mx)=\lim_{x\to0^+}\frac{m-m^2}{x(m+1)}=\begin{cases}\infty & \text{if } 0<m<1\\ -\infty & \text{if } m<0 \text{ or } m>1\end{cases} \quad \text{case } \tfrac{b}{0}$$

Ergo, $\lim\limits_{(x,y)\to(0,0)}\dfrac{x^2y-xy^2}{x^4+y^4} \nexists$

16. Show that $f(x,y)=x^3-2x^2y+y^2$ is differentiable at $(2,1)$ using the definition with ε.

$f_x(x,y)=3x^2-4xy$ $\quad f_x(2,1)=4$

$f_y(x,y)=-2x^2+2y$ $\quad f_y(2,1)=-6$

Hence we have: $f_x(2,1)\Delta x+f_y(2,1)\Delta y=4\Delta x-6\Delta y$

$$\begin{aligned}\Delta f &= f(2+\Delta x,1+\Delta y)-f(2,1)\\ &=(2+\Delta x)^3-2(2+\Delta x)^2(1+\Delta y)+(1+\Delta y)^2-1\\ &=8+12\Delta x+6\Delta x^2+\Delta x^3-8-8\Delta x-2\Delta x^2-8\Delta y-8\Delta x\Delta y-2\Delta x^2\Delta y+1+2\Delta y+\Delta y^2-1\\ &=4\Delta x-6\Delta y+4\Delta x^2+\Delta x^3-8\Delta x\Delta y-2\Delta x^2\Delta y+\Delta y^2\\ &=f_x(2,1)\Delta x+f_y(2,3)1\Delta y+(4\Delta x+\Delta x^2)\Delta x+(-8\Delta x-2\Delta x^2+\Delta y)\Delta y\\ &=f_x(2,3)\Delta x+f_y(2,3)\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y\end{aligned}$$

Thus if $\varepsilon_1=4\Delta x+\Delta x^2$ and $\varepsilon_2=-8\Delta x-2\Delta x^2+\Delta y$, then

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_1 = \lim_{(\Delta x,\Delta y)\to(0,0)} \left(4\Delta x + \Delta x^2\right) = 0$$
$$\lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_2 = \lim_{(\Delta x,\Delta y)\to(0,0)} \left(-8\Delta x - 2\Delta x^2 + \Delta y\right) = 0$$

Ergo, f is differentiable at $(2,1)$.

26. If f is a differentiable function, show that the function $w = x^2 f\left(\frac{y}{x}\right)$ satisfies $x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 2w$.

Let $u = x^2$ and $v = \frac{y}{x}$, then $w = uf(v)$.

$$\begin{aligned} x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} &= x\left(\frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}\right) + y\left(\frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}\right) \\ &= x\left(f(v)2x + uf'(v)\left(\tfrac{-1}{x^2}\right)\right) + yuf'(v)\tfrac{1}{x} \\ &= 2x^2 f(v) \\ &= 2w \end{aligned}$$

27. If f and g are twice differentiable functions, show that the function $w = xf(x+y) + yg(x+y)$ satisfies $\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 w}{\partial x\partial y} + \frac{\partial^2 w}{\partial y^2} = 0$.

Let $t = x,\ u = x + y,\ v = y$, then $w = tf(u) + vg(u)$.

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = f(u) + tf'(u) + vg'(u)$$
$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial y} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} = tf'(u) + vg'(u) + g(u)$$
$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} &= \frac{\partial}{\partial t}\left[f(u) + tf'(u) + vg'(u)\right]\frac{\partial t}{\partial x} + \frac{\partial}{\partial u}\left[f(u) + tf'(u) + vg'(u)\right]\frac{\partial u}{\partial x} \\ &\quad + \frac{\partial}{\partial v}\left[f(u) + tf'(u) + vg'(u)\right]\frac{\partial v}{\partial x} \\ &= f'(u) + f'(u) + tf''(u) + vg''(u) + g'(u) \end{aligned}$$
$$\begin{aligned} \frac{\partial^2 w}{\partial y\partial x} &= \frac{\partial}{\partial t}\left[f(u) + tf'(u) + vg'(u)\right]\frac{\partial t}{\partial y} + \frac{\partial}{\partial u}\left[f(u) + tf'(u) + vg'(u)\right]\frac{\partial u}{\partial y} \\ &\quad + \frac{\partial}{\partial v}\left[f(u) + tf'(u) + vg'(u)\right]\frac{\partial v}{\partial y} \\ &= f'(u) + tf''(u) + vg''(u) + g'(u) \end{aligned}$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial t}\left[tf'(u)+vg'(u)+g(u)\right]\frac{\partial t}{\partial y} + \frac{\partial}{\partial u}\left[tf'(u)+vg'(u)+g(u)\right]\frac{\partial u}{\partial y}$$
$$+\frac{\partial}{\partial v}\left[tf'(u)+vg'(u)+g(u)\right]\frac{\partial v}{\partial y}$$
$$= tf''(u)+vg''(u)+g'(u)+g'(u)$$
$$\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2} = f'(u)+f'(u)+tf''(u)+vg''(u)+g'(u)-2f'(u)-2tf''(u)$$
$$-2vg''(u)-2g'(u)+tf''(u)+vg''(u)+g'(u)+g'(u)$$
$$=0$$

28. If f has continuous partial derivatives, show that the function $w = f\left(x^2-y^2, y^2-x^2\right)$ satisfies $y\frac{\partial w}{\partial x} + x\frac{\partial w}{\partial y} = 0$.

Let $u = x^2 - y^2$, $v = y^2 - x^2$, then $w = f(u,v)$

$$y\frac{\partial w}{\partial x} + x\frac{\partial w}{\partial y} = y\left(\frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}\right) + x\left(\frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}\right)$$
$$= 2xy\frac{\partial w}{\partial u} - 2xy\frac{\partial w}{\partial v} - 2xy\frac{\partial w}{\partial u} + 2xy\frac{\partial w}{\partial v}$$
$$=0$$

29. If f is a differentiable function of two variables and $w = f(x,y)$, where $\begin{cases} x = u^2 - v^2 \\ y = v^2 - u^2 \end{cases}$ then prove that $\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2} = \frac{u^2+v^2}{-uv}\frac{\partial^2 w}{\partial v \partial u}$.

$$\frac{\partial w}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = 2u\frac{\partial f}{\partial x} - 2u\frac{\partial f}{\partial y} = 2u\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right)$$
$$\frac{\partial^2 w}{\partial u^2} = \frac{\partial}{\partial u}[2u]\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) + 2u\frac{\partial}{\partial u}\left[\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right]$$
$$= 2\frac{\partial f}{\partial x} - 2\frac{\partial f}{\partial y} + 2u\frac{\partial}{\partial u}\left[\frac{\partial f}{\partial x}\right] - 2u\frac{\partial}{\partial u}\left[\frac{\partial f}{\partial y}\right]$$
$$= 2\frac{\partial f}{\partial x} - 2\frac{\partial f}{\partial y} + 2u\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial u} + 2u\frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial u} - 2u\frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial u} - 2u\frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial u}$$
$$= 2\frac{\partial f}{\partial x} - 2\frac{\partial f}{\partial y} + 4u^2\frac{\partial^2 f}{\partial x^2} - 4u^2\frac{\partial^2 f}{\partial y \partial x} + 4u^2\frac{\partial^2 f}{\partial y^2}$$
$$\frac{\partial w}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = -2v\frac{\partial f}{\partial x} + 2v\frac{\partial f}{\partial y} = 2v\left(-\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 w}{\partial v^2}=\frac{\partial}{\partial v}[2v]\left(-\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\right)+2v\frac{\partial}{\partial v}\left[\left(-\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\right)\right]$$

$$=-2\frac{\partial f}{\partial x}+2\frac{\partial f}{\partial y}-2v\frac{\partial}{\partial v}\left[\frac{\partial f}{\partial x}\right]+2v\frac{\partial}{\partial v}\left[\frac{\partial f}{\partial y}\right]$$

$$=-2\frac{\partial f}{\partial x}+2\frac{\partial f}{\partial y}-2v\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial v}-2v\frac{\partial^2 f}{\partial y\partial x}\frac{\partial y}{\partial v}+2v\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial v}+2v\frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial v}$$

$$=-2\frac{\partial f}{\partial x}+2\frac{\partial f}{\partial y}+4v^2\frac{\partial^2 f}{\partial x^2}-4v^2\frac{\partial^2 f}{\partial y\partial x}+4v^2\frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 w}{\partial u^2}+\frac{\partial^2 w}{\partial v^2}=4\left(u^2+v^2\right)\frac{\partial^2 f}{\partial x^2}-4\left(u^2+v^2\right)\frac{\partial^2 f}{\partial y\partial x}+4\left(u^2+v^2\right)\frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 w}{\partial v\partial u}=2u\frac{\partial}{\partial v}\left[\frac{\partial f}{\partial x}-\frac{\partial f}{\partial y}\right]$$

$$=2u\frac{\partial}{\partial v}\left[\frac{\partial f}{\partial x}\right]-2u\frac{\partial}{\partial v}\left[\frac{\partial f}{\partial y}\right]$$

$$=2u\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial v}+2u\frac{\partial^2 f}{\partial y\partial x}\frac{\partial y}{\partial v}-2u\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial v}-2u\frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial v}$$

$$=-4uv\frac{\partial^2 f}{\partial x^2}-4uv\frac{\partial^2 f}{\partial y\partial x}-4uv\frac{\partial^2 f}{\partial y^2}$$

$$\frac{u^2+v^2}{-uv}\frac{\partial^2 w}{\partial v\partial u}=4\left(u^2+v^2\right)\frac{\partial^2 f}{\partial x^2}-4\left(u^2+v^2\right)\frac{\partial^2 f}{\partial y\partial x}+4\left(u^2+v^2\right)\frac{\partial^2 f}{\partial y^2}$$

Hence $\frac{\partial^2 w}{\partial u^2}+\frac{\partial^2 w}{\partial v^2}=\frac{u^2+v^2}{-uv}\frac{\partial^2 w}{\partial v\partial u}$

44. Calculate the iterated integral.

a) $\int_0^1\int_{x^2}^1 x\sqrt{x^2+y}\,dydx=\int_0^1\int_0^{\sqrt{y}} x\sqrt{x^2+y}\,dxdy=\int_0^1\left(\frac{2\sqrt{2}}{3}y^{\frac{3}{2}}-\frac{1}{3}y^{\frac{3}{2}}\right)dy=\frac{4\sqrt{2}}{15}-\frac{2}{15}$

b) $\int_{-2}^2\int_0^{\sqrt{4-x^2}}\sqrt{x^2+y^2}\,dydx=\int_0^{\pi}\int_0^2 r^2\,drd\theta=\frac{8\pi}{3}$

c) $\int_0^1\int_0^x\frac{x}{\sqrt{1-y^2}}\,dydx=\int_0^1 x\arcsin x\,dx=\frac{\pi}{8}$

d) $\int_0^8\int_{\frac{y}{4}}^2\sin\left(x^2\right)dxdy=\int_0^2\int_0^{4x}\sin\left(x^2\right)dydx=\int_0^2 4x\sin\left(x^2\right)dx=2-2\cos 64$

e) $\int_0^4\int_{-\sqrt{4x-x^2}}^{\sqrt{4x-x^2}}\frac{xye^{x^2+y^2}}{x^2+y^2}\,dydx=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_0^{4\cos\theta} r\cos\theta\sin\theta e^{r^2}\,drd\theta=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_0^{4\cos\theta} r\cos\theta\sin\theta e^{r^2}\,drd\theta$

$$=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\left(\tfrac{1}{2}\cos\theta\sin\theta e^{16\cos^2\theta}-\tfrac{1}{2}\cos\theta\sin\theta\right)d\theta=0$$

f) $\int_0^1\int_0^{2-x}\int_2^{6-x-y^2} xdzdydx = \int_0^1\int_0^{2-x}\left(4-x-y^2\right)dydx = \frac{1}{3}\int_0^1\left(x^4-3x^3-6x^2+16x\right)dx = \frac{109}{60}$

g) $\int_0^1\int_0^1\int_{x^2}^1 12xze^{zy^2}dydxdz = \int_0^1\int_0^1\int_0^{\sqrt{y}} 12xze^{zy^2}dxdydz = \int_0^1\int_0^1 6yze^{zy^2}dydz$

$$= \int_0^1\left(3e^z-3\right)dz = 3e-6$$

h) $\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_0^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\left(y^2+z^2\right)dxdydz = \int_0^{\pi}\int_0^3\int_{\frac{-1}{2}}^{\frac{1}{2}}\left(r^2\sin^2\theta+z^2\right)rdzdrd\theta$

$$= \int_0^{\pi}\int_0^3\left(r^3\sin^2\theta+\tfrac{1}{12}r\right)drd\theta$$

$$= \int_0^{\pi}\left(\tfrac{81}{4}\sin^2\theta+\tfrac{3}{8}\right)d\theta = \tfrac{21\pi}{2}$$

i) $\int_{-2}^2\int_0^{\sqrt{4-x^2}}\int_{-\sqrt{4-x^2-y^2}+2}^{\sqrt{4-x^2-y^2}+2}\frac{1}{\sqrt{x^2+y^2+z^2}}dzdydx = \int_0^{\pi}\int_0^{\frac{\pi}{2}}\int_0^{4\cos\phi}\rho\sin\phi d\rho d\phi d\theta$

$$= \int_0^{\pi}\int_0^{\frac{\pi}{2}} 8\cos^2\phi\sin\phi d\phi d\theta$$

$$= \int_0^{\pi}\tfrac{8}{3}d\theta = \tfrac{8\pi}{3}$$

j) $\int_0^1\int_0^{\sqrt{x-x^2}}\int_{x^2+y^2}^{9-x^2-y^2} xyz^2dzdydx = \int_0^{\frac{\pi}{2}}\int_0^{\cos\theta}\int_{r^2}^{9-r^2} r^3\cos\theta\sin\theta z^2dzdrd\theta$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\cos\theta}\tfrac{1}{3}\left(-2r^9+27r^7-243r^5+729r^3\right)\cos\theta\sin\theta drd\theta$$

$$= \int_0^{\frac{\pi}{2}}\left(\tfrac{-1}{15}\cos^{11}\theta+\tfrac{9}{8}\cos^9\theta-\tfrac{27}{2}\cos^7\theta+\tfrac{243}{5}\cos^5\theta\right)\sin\theta d\theta$$

$$= \tfrac{769}{90}$$

45. Evaluate the double or triple integral.

a) $\iint_R\left(x^2y-y^2\right)dA$ where R is the region bounded by $y=4-x^2$ and $y=x+2$.

$$\iint_R\left(x^2y-y^2\right)dA = \int_{-2}^1\int_{x+2}^{4-x^2}\left(x^2y-y^2\right)dydx = \ldots = \tfrac{-3429}{140}$$

b) $\iint_R\frac{1}{\sqrt{x^2+y^2}}dA$ where R is the region outside $x^2+y^2=4$ and inside $x^2+y^2=4x$.

$$\iint_R\tfrac{1}{\sqrt{x^2+y^2}}dA = \int_{\frac{-\pi}{3}}^{\frac{\pi}{3}}\int_2^{4\cos\theta}drd\theta = \ldots = 4\sqrt{3}-\tfrac{4}{3}\pi$$

c) $\iint_R\frac{x}{\sqrt{1+x^2+y^2}}dA$ where R is the region bounded by $y=\frac{1}{2}x^2$, $x=2$ and $y=0$.

$$\iint_R\tfrac{x}{\sqrt{1+x^2+y^2}}dA = \int_0^2\int_{\sqrt{2y}}^2\tfrac{x}{\sqrt{1+x^2+y^2}}dxdy = \ldots = \tfrac{5}{4}\ln 4-1$$

d) $\iint_R\frac{x}{y}dydx$ where R is the region bounded below by the line $y=1$ and above by the circle $x^2+y^2=4$.

$$\iint_R \tfrac{x}{y}\,dydx = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{\sec\theta}^{2} r\cot\theta\, dr = \ldots = 0$$

e) $\iint_R \frac{1}{\sqrt{x^2+y^2}}\,dA$ where R is the region to the right of $x=1$ bounded by $x=1$, $y=0$ and $\sqrt{2x-x^2}$.

$$\iint_R \tfrac{1}{\sqrt{x^2+y^2}}\,dA = \int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{2\cos\theta} dr = \ldots = \sqrt{2} + \tfrac{1}{2}\ln 2 - \ln\left(\sqrt{2}+2\right)$$

f) $\iiint_S xyz\,dV$ where S is bounded below by $z = \frac{x^2}{9} + \frac{y^2}{16}$ and above by $z=1$.

$$\iiint_S xyz\,dV = \int_{-3}^{3} \int_{-4\sqrt{1-\frac{x^2}{9}}}^{-4\sqrt{1-\frac{x^2}{9}}} \int_{\frac{x^2}{9}+\frac{y^2}{16}}^{1} xyz\,dz = \ldots = 0$$

g) $\iiint_S \left(x^2y - z\right)dV$ where S is bounded by $y+z=1$, $z=2y$, $z=y$, $x=0$ and $x=3$.

$$\iiint_S \left(x^2y-z\right)dV = \int_0^{\frac{1}{3}} \int_y^{2y} \int_0^3 \left(x^2y-z\right)dxdzdy + \int_{\frac{1}{3}}^{\frac{1}{2}} \int_y^{1-y} \int_0^3 \left(x^2y-z\right)dxdzdy = \ldots = \tfrac{1}{9}$$

h) $\iiint_S \frac{z}{\sqrt{x^2+y^2}}\,dV$ where S is bounded by $z=xy$, $x^2+y^2=4$ and $z=0$.

$$\iiint_S \frac{z}{\sqrt{x^2+y^2}}\,dV = \int_0^{2\pi} \int_0^2 \int_0^{r^2\sin\theta\cos\theta} z\,dzdrd\theta = \ldots = \tfrac{4\pi}{5}$$

i) $\iiint_S y\,dV$ where S is bounded by $z = 2\sqrt{x^2+y^2}$, $x^2+y^2=4$ and $z=0$.

$$\iiint_S y\,dV = \int_0^{2\pi} \int_0^2 \int_0^{2r} r^2\sin\theta\,dzdrd\theta = \ldots = 0$$

j) $\iiint_S x\,dV$ where S is bounded by $z=\sqrt{x^2+y^2}$, $z = x^2+y^2-4$ and $x^2+y^2=2y$.

$$\iiint_S y\,dV = \int_0^{\pi} \int_0^{2\sin\theta} \int_r^{r^2-4} r^2\sin\theta\,dzdrd\theta = \ldots = 0$$

47. Determine if the following integrals converge. If so, to what value?

a) $\iint_{\mathbb{R}^2} \frac{1}{x^2+y^2}\,dA$

$$\iint_{\mathbb{R}^2} \frac{1}{x^2+y^2}\,dA = \int_0^{2\pi} \int_0^{\infty} \tfrac{1}{r^2} r\,drd\theta = \lim_{t\to\infty} \int_0^{2\pi} \int_0^{t} \tfrac{1}{r^2} r\,drd\theta = \ldots = \infty$$

b) $\iint_R \frac{x}{x^2+y^2}\,dA$ where R is the unit square in the first quadrant.

$$\iint_R \frac{x}{x^2+y^2}\,dA = \lim_{t\to 0^+} \int_t^1 \int_0^1 \frac{x}{x^2+y^2}\,dydx = \ldots = \tfrac{1}{2}\ln 2 + \tfrac{\pi}{4}$$

c) $\iiint\limits_S \frac{1}{x^2+x^4+x^2y^2+x^2z^2}\,dV$ where S is the inside of the cone $z=\sqrt{x^2+y^2}$

$$\iiint\limits_S \frac{1}{x^2+x^4+x^2y^2+x^2z^2}\,dV = \iiint\limits_S \frac{1}{x^2\left(1+x^2+y^2+z^2\right)}\,dV$$
$$= \lim_{t\to\infty}\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^t \frac{1}{\rho^2\cos^2\theta\sin^2\phi\left(1+\rho^2\right)}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$= \ldots = \left(\sqrt{2}-1\right)\pi^2$$

48. Find the centroid of the given region R.

a) R is the region bounded by $y=x^3-x$ and $y=3x$ in the first and fourth quadrants.

$$A=\iint\limits_R dA=\int_0^2\int_{x^3-x}^{3x} dydx=\ldots=4 \qquad \bar{x}=\tfrac{1}{A}\iint\limits_R xdA=\tfrac{1}{4}\int_0^2\int_{x^3-x}^{3x} xdydx=\ldots=\tfrac{16}{15}$$
$$\bar{y}=\tfrac{1}{A}\iint\limits_R ydA=\tfrac{1}{4}\int_0^2\int_{x^3-x}^{3x} ydydx=\ldots=\tfrac{208}{105}$$

b) R is the region inside the cardiod $r=1+\sin\theta$ and outside the circle $r=1$.

$$A=\iint\limits_R dA=\int_0^{\pi}\int_1^{1+\sin\theta} rdrd\theta=\ldots=\tfrac{\pi}{4}+2$$
$$\bar{x}=\tfrac{1}{A}\iint\limits_R xdA=\tfrac{4}{\pi+8}\int_0^{\pi}\int_1^{1+\sin\theta} r^2\cos\theta drd\theta=\ldots=0$$
$$\bar{y}=\tfrac{1}{A}\iint\limits_R ydA=\tfrac{4}{\pi+8}\int_0^{\pi}\int_1^{1+\sin\theta} r^2\sin\theta drd\theta=\ldots=\tfrac{15\pi+32}{6\pi+48}$$

c) R is the triangle with vertices $(1,1)$, $(2,2)$ and $(3,1)$.

$$A=\iint\limits_R dA=\int_1^2\int_y^{4-y} dxdy=\ldots=1 \qquad \bar{x}=\tfrac{1}{A}\iint\limits_R xdA=\int_1^2\int_y^{4-y} xdxdy=\ldots=2$$
$$\bar{y}=\tfrac{1}{A}\iint\limits_R ydA=\int_1^2\int_y^{4-y} ydxdy=\ldots=\tfrac{4}{3}$$

49. Find the center of mass and the moments of inertia for the region in the first quadrant bounded by $y=1-x$ and $x^2+y^2=1$ if the density is given by $\delta(x,y)=x+y$.

$$M=\iint\limits_R \delta(x,y)\,dA=\int_0^1\int_{1-x}^{\sqrt{1-x^2}}(x+y)\,dydx=\ldots=\tfrac{1}{3}$$
$$\bar{x}=\tfrac{1}{M}\iint\limits_R x\delta(x,y)\,dA=3\int_0^1\int_{1-x}^{\sqrt{1-x^2}} x(x+y)\,dydx=\ldots=\tfrac{3\pi}{16}$$
$$\bar{y}=\tfrac{1}{M}\iint\limits_R y\delta(x,y)\,dA=3\int_0^1\int_{1-x}^{\sqrt{1-x^2}} y(x+y)\,dydx=\ldots=\tfrac{3\pi}{16}$$

$$I_x = \frac{1}{M}\iint_R y^2\delta(x,y)\,dA = 3\int_0^1\int_{1-x}^{\sqrt{1-x^2}} y^2(x+y)\,dydx = \ldots = \frac{2}{15}$$

$$I_y = \frac{1}{M}\iint_R x^2\delta(x,y)\,dA = 3\int_0^1\int_{1-x}^{\sqrt{1-x^2}} x^2(x+y)\,dydx = \ldots = \frac{2}{15}$$

50. Find the volume of each solid.

a) The solid bounded by the paraboloids $z = x^2 + 2y^2$ and $z = 12 - 2x^2 - y^2$.

$$V = \iiint_S dV = \int_0^{2\pi}\int_0^2\int_{r^2+2r^2\sin^2\theta}^{12-r^2-r^2\cos^2\theta} rdzdrd\theta = \ldots = 24\pi$$

b) The solid bounded by $y = z^2$, $z = y^2$, $x + y + z = 2$ and $x = 0$.

$$V = \iiint_S dV = \int_0^1\int_{y^2}^{\sqrt{y}}\int_0^{2-y-z} dxdzdy = \ldots = \frac{11}{30}$$

c) The solid bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 2y$.

$$V = \iiint_S dV = \int_0^{\pi}\int_0^{2\sin\theta}\int_{r^2}^{2r\sin\theta} rdzdrd\theta = \ldots = \frac{\pi}{2}$$

d) The solid inside $x^2 + y^2 + z^2 = 12$ and outside $z^2 = 3x^2 + 3y^2$.

$$V = \iiint_S dV = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^{\sqrt{12}} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \ldots = 24\pi$$

e) The solid bounded by $(x^2 + y^2 + z^2)^2 = x$.

$$V = \iiint_S dV = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{\sin^{\frac{1}{3}}\phi\cos^{\frac{1}{3}}\theta} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \ldots = \frac{\pi}{3}$$

f) The solid bounded by $z = 1 - x^2$, $z = x^2 - 1$, $y + z = 1$ and $y = 0$.

$$V = \iiint_S dV = \int_{-1}^1\int_{x^2-1}^{1-x^2}\int_0^{1-z} dydzdx = \ldots = \frac{8}{3}$$

51. Find the centroid of the given solid *S*.

a) *S* is bounded by the paraboloid $z = b(x^2 + y^2)$ and the plane $z = h$ where $b > 0$ and $h > 0$.

$$V = \iiint_S dV = \int_0^{2\pi}\int_0^{\frac{\sqrt{h}}{\sqrt{b}}}\int_{br^2}^{h} rdzdrd\theta = \ldots = \frac{\pi h^2}{2b}$$

By symmetry, $\bar{x} = \bar{y} = 0$

$$\bar{z} = \frac{1}{V}\iiint_S zdV = \frac{2b}{h^2\pi}\int_0^{2\pi}\int_0^{\frac{\sqrt{h}}{\sqrt{b}}}\int_{br^2}^{h} rzdzdrd\theta = \ldots = \frac{2}{3}h$$

b) *S* is bounded by the coordinate planes $y + z = 2$ and $x = 3$.

$$V = \iiint_S dV = \int_0^3\int_0^2\int_0^{2-y} dzdydx = \ldots = 6$$

$$\bar{x} = \tfrac{1}{V}\iiint_S x\,dV = \tfrac{1}{6}\int_0^3\int_0^2\int_0^{2-y} x\,dz\,dy\,dx = \ldots = \tfrac{3}{2}$$

$$\bar{y} = \tfrac{1}{V}\iiint_S y\,dV = \tfrac{1}{6}\int_0^3\int_0^2\int_0^{2-y} y\,dz\,dy\,dx = \ldots = \tfrac{2}{3}$$

$$\bar{z} = \tfrac{1}{V}\iiint_S z\,dV = \tfrac{1}{6}\int_0^3\int_0^2\int_0^{2-y} z\,dz\,dy\,dx = \ldots = \tfrac{2}{3}$$

c) S is bounded above by $x^2 + y^2 + z^2 = 4z$ and below by $z = 1$.

$$V = \iiint_S dV = \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_{\sec\phi}^{4\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \ldots = 9\pi$$

By symmetry, $\bar{x} = \bar{y} = 0$

$$\bar{z} = \tfrac{1}{V}\iiint_S z\,dV = \tfrac{1}{9\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_{\sec\phi}^{4\cos\phi} \rho^3 \sin\phi\cos\phi\, d\rho\, d\phi\, d\theta = \ldots = \tfrac{9}{4}$$

52. Find the center of mass and the moments of inertia for the solid S.

a) S is between the cone $x^2 + y^2 = z^2$ and the paraboloid $z = \tfrac{1}{4}x^2 + \tfrac{1}{4}y^2$ if the density is given by $\delta(x,y,z) = \sqrt{x^2 + y^2}$.

$$M = \iiint_S \delta(x,y,z)\,dV = \int_0^{2\pi}\int_0^2\int_{\frac{1}{4}r^2}^{r} r^2\,dz\,dr\,d\theta = \ldots = \tfrac{24\pi}{5}$$

By symmetry, $\bar{x} = \bar{y} = 0$

$$\bar{z} = \tfrac{1}{M}\iiint_S z\delta(x,y,z)\,dV = \tfrac{5}{24\pi}\int_0^{2\pi}\int_0^2\int_{\frac{1}{4}r^2}^{r} r^2 z\,dz\,dr\,d\theta = \ldots = \tfrac{23}{21}$$

$$I_x = \iiint_S (y^2 + z^2)\delta(x,y,z)\,dV = \int_0^{2\pi}\int_0^2\int_{\frac{1}{4}r^2}^{r} r^2(r^2\sin\theta + z^2)\,dz\,dr\,d\theta = \ldots = \tfrac{2384\pi}{189}$$

$$I_y = \iiint_S (x^2 + z^2)\delta(x,y,z)\,dV = \int_0^{2\pi}\int_0^2\int_{\frac{1}{4}r^2}^{r} r^2(r^2\cos\theta + z^2)\,dz\,dr\,d\theta = \ldots = \tfrac{2384\pi}{189}$$

$$I_z = \iiint_S (x^2 + y^2)\delta(x,y,z)\,dV = \int_0^{2\pi}\int_0^2\int_{\frac{1}{4}r^2}^{r} r^4\,dz\,dr\,d\theta = \ldots = \tfrac{256\pi}{21}$$

b) S is the solid in the first octant bounded by the surface $x^2 + y = z$ and the planes $y = x$, $x = 1$ if the density is given by $\delta(x,y,z) = 1 + xyz$.

$$M = \iiint_S \delta(x,y,z)\,dV = \int_0^1\int_0^x\int_0^{x^2+y}(1 + xyz)\,dz\,dy\,dx = \ldots = \tfrac{347}{672}$$

$$\bar{x} = \tfrac{1}{M}\iiint_S x\delta(x,y,z)\,dV = \tfrac{672}{347}\int_0^1\int_0^x\int_0^{x^2+y}(x + x^2yz)\,dz\,dy\,dx = \ldots = \tfrac{4156}{5205}$$

$$\bar{y} = \tfrac{1}{M}\iiint_S y\delta(x,y,z)\,dV = \tfrac{672}{347}\int_0^1\int_0^x\int_0^{x^2+y}(y + xy^2z)\,dz\,dy\,dx = \ldots = \tfrac{7481}{15615}$$

$$\bar{z} = \tfrac{1}{M}\iiint_S z\delta(x,y,z)\,dV = \tfrac{672}{347}\int_0^1\int_0^x\int_0^{x^2+y}(z + xyz^2)\,dz\,dy\,dx = \ldots = \tfrac{9661}{15615}$$

$$I_x = \iiint_S \left(y^2+z^2\right)\delta(x,y,z)\,dV = \int_0^1\int_0^x\int_0^{x^2+y}\left(y^2+z^2\right)(1+xyz)\,dzdydx = \ldots = \tfrac{97943}{221760}$$

$$I_y = \iiint_S \left(x^2+z^2\right)\delta(x,y,z)\,dV = \int_0^1\int_0^x\int_0^{x^2+y}\left(x^2+z^2\right)(1+xyz)\,dzdydx = \ldots = \tfrac{42265}{66528}$$

$$I_z = \iiint_S \left(x^2+y^2\right)\delta(x,y,z)\,dV = \int_0^1\int_0^x\int_0^{x^2+y}\left(x^2+y^2\right)(1+xyz)\,dzdydx = \ldots = \tfrac{4277}{8640}$$

c) S is the solid given by the equation $x^2+y^2+z^2=6z$ and if the density is given by $\delta(x,y,z)=x^2+y^2+z^2+1$.

$$M = \iiint_S \delta(x,y,z)\,dV = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{6\cos\phi}\left(\rho^2+1\right)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \ldots = \tfrac{2772\pi}{5}$$

By symmetry, $\overline{x}=\overline{y}=0$

$$\overline{z} = \tfrac{1}{M}\iiint_S z\delta(x,y,z)\,dV = \tfrac{2772\pi}{5}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{6\cos\phi}\rho\cos\phi\left(\rho^2+1\right)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \ldots = \tfrac{287}{77}$$

$$I_x = \iiint_S \left(y^2+z^2\right)\delta(x,y,z)\,dV$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{6\cos\phi}\left(\rho^2\sin^2\phi\sin^2\theta+\rho^2\cos^2\phi\right)\left(\rho^2+1\right)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \ldots = \tfrac{330804\pi}{35}$$

$$I_y = \iiint_S \left(x^2+z^2\right)\delta(x,y,z)\,dV$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{6\cos\phi}\left(\rho^2\sin^2\phi\cos^2\theta+\rho^2\cos^2\phi\right)\left(\rho^2+1\right)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \ldots = \tfrac{330804\pi}{35}$$

$$I_z = \iiint_S \left(x^2+y^2\right)\delta(x,y,z)\,dV$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{6\cos\phi}\rho^2\sin^2\phi\left(\rho^2+1\right)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \ldots = \tfrac{14904\pi}{7}$$

54. Find the area of the given surface.

a) The portion of the cone $z=\sqrt{x^2+y^2}$ that lies inside the cylinder $x^2+y^2=2x$.

$$\vec{R}(r,\theta)=(r\cos\theta, r\sin\theta, r) \qquad \frac{\partial\vec{R}}{\partial r}\times\frac{\partial\vec{R}}{\partial\theta}=(-r\cos\theta,-r\sin\theta,r)$$

$$S=\iint_R\left\|\frac{\partial\vec{r}}{\partial r}\times\frac{\partial\vec{r}}{\partial\theta}\right\|dA=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta}\sqrt{2}r\,dr\,d\theta=\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}}2\sqrt{2}\cos^2\theta\, d\theta=\sqrt{2}\pi$$

b) The portion of the surface $z=2x^2+\sqrt{3}y$ that lies above the triangular region with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

$$\vec{r}(u,v)=\left(u,v,2u^2+\sqrt{3}v\right) \qquad \frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}=\left(-4u,-\sqrt{3},1\right)$$

$$S=\iint_R\left\|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right\|dA=\int_0^1\int_0^u 2\sqrt{4u^2+1}\,dv\,du=\int_0^1 2u\sqrt{4u^2+1}\,du=\tfrac{5\sqrt{5}}{6}-\tfrac{1}{6}$$

48. Using a change of variables, evaluate the integral.

a) $\iint_R \left(y^2 - x^2\right) dA$ where R is the region in the first quadrant bounded by $y - x = 0$, $y - x = 1$, $xy = 1$ and $xy = 2$;

$u = y - x$

$v = xy$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} -1 & 1 \\ y & x \end{vmatrix}} = \frac{-1}{x+y}$$

$$\iint_R \left(y^2 - x^2\right) dA = \int_1^2 \int_0^1 u\,du\,dv = \tfrac{1}{2}$$

b) $\iiint_S dV$ where S is the solid bounded by the planes $x + y = 1$, $x + y = 2$, $3y - x = 0$, $3y - x = 6$, $x + y + z = 5$ and $x + y + z = 10$;

$u = x + y$ $\qquad x = \tfrac{3}{4}u - \tfrac{1}{4}v$

$v = 3y - x$ $\qquad y = \tfrac{1}{4}u + \tfrac{1}{4}v$

$w = x + y + z$ $\qquad z = w - v$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{3}{4} & \frac{-1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & -1 & 1 \end{vmatrix} = \tfrac{1}{4} \qquad \iiint_S dV = \int_1^2 \int_0^6 \int_5^{10} \tfrac{1}{4}\,dw\,dv\,du = \tfrac{15}{2}$$

58. Solve the following differential equations.

a) $x\dfrac{dy}{dx} + 2y = 5x^4$ $\qquad \dfrac{dy}{dx} + \dfrac{2y}{x} = 5x^3$ $\qquad$ Linear

b) $\dfrac{d^2y}{dx^2} = e^y \dfrac{dy}{dx}$ $\qquad$ x-missing, let $p = \dfrac{dy}{dx}$, then $\dfrac{d^2y}{dx^2} = p\dfrac{dp}{dy}$, $p\dfrac{dp}{dy} = pe^y$, seperable

c) $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$ $\qquad (x+y)dx + (y-x)dy = 0$ $\qquad$ Homogenous, let $y = ux$

d) $\left(2xy - 9x^2\right)dx + \left(2y + x^2 + 1\right)dy = 0$ $\qquad \frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$ $\qquad$ exact

e) $\dfrac{d^2y}{dx^2} - 10\dfrac{dy}{dx} + 41y = x + e^x$ $\qquad$ characteristic equation: $m^2 - 10m + 41 = 0$

$y_h = K_1 e^{5x}\cos 4x + K_2 e^{5x}\sin 4x$ $\qquad y_p = Ax + B + Ce^x$

f) $\left(e^{x+y} + ye^y\right)dx + \left(xe^y - 1\right)dy = 0$ $\qquad \dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = -1$ $\qquad$ Almost exact

g) $x\dfrac{dy}{dx} - 2y + x^3y^2 = 0$ $\qquad \dfrac{dy}{dx} - \dfrac{2y}{x} = -x^2y^2$ $\qquad$ Bernoulli

h) $\left(x\sec\frac{y}{x} + y\right)dx - x\,dy = 0$ $\qquad$ Homogenous

i) $xydx + e^{-x^2}\left(y^2 - 1\right)dy = 0$ $\quad xe^{x^2}dx + \frac{y^2-1}{y}dy = 0$ $\quad$ Seperable

j) $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx} + 1\right)^2$ $\quad$ y-missing, let $p = \dfrac{dy}{dx}$, then $\dfrac{dp}{dx} = (p+1)^2$

k) $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = e^x + \sin x$ $\quad$ characteristic equation: $m^3 - 3m^2 + 3m - 1 = 0$

$y_h = K_1e^x + K_2xe^x + K_3x^2e^x$ $\quad$ $y_p = Ax^3e^x + B\sin x + C\cos x$

l) $x\dfrac{d^2y}{dx^2} = \cot\left(\dfrac{dy}{dx}\right)$ $\quad$ y-missing, let $p = \dfrac{dy}{dx}$, then $x\dfrac{dp}{dx} = \cot p$, seperable

m) $\left(\sin y\cos y + x\cos^2 y\right)dx + xdy = 0$ $\quad$ $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = 2\tan y$, almost exact

n) $2x\dfrac{dy}{dx} - y = x + 1$ $\quad$ $\dfrac{dy}{dx} - \dfrac{y}{2x} = \dfrac{1}{2} + \dfrac{1}{2x}$ $\quad$ Linear

o) $\left(y^2 - e^{-2x}\right)dx + xydy = 0$ $\quad$ $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{1}{x}$, almost exact

p) $x\dfrac{dy}{dx} + 2y + x^3y^2\cos x = 0$ $\quad$ $\dfrac{dy}{dx} + \dfrac{2y}{x} = -x^2y^2\cos x$ $\quad$ Bernoulli