MATHEMATICS 201-BNK-05
Advanced Calculus
Martin Huard
Winter 2011

IX - Directional Derivatives and Gradients

1. Find the directional derivative of f at the given point in the direction indicated by the angle θ.
   a) $f(x,y)=x^3y-x^2y^2$ $\quad (2,1)$ $\quad \theta=\frac{\pi}{3}$
   b) $f(x,y)=xe^{xy+6}-y^2$ $\quad (2,-3)$ $\quad \theta=\frac{-\pi}{6}$
   c) $f(x,y)=\sin x\cos y-\sin(\cos y)$ $\quad \left(\frac{\pi}{4},\frac{\pi}{2}\right)$ $\quad \theta=\frac{\pi}{2}$

2. For each problem, find the gradient, the gradient evaluate at the given point, and the directional derivative at the given point in the direction of the vector $\vec{u}$.
   a) $f(x,y)=\ln\left(x^2+y^2\right)$ $\quad (3,4)$ $\quad \vec{u}=\left(\frac{5}{13},\frac{-12}{13}\right)$
   b) $f(x,y)=\sqrt{x^2-y^2}$ $\quad (5,-3)$ $\quad \vec{u}=(1,2)$
   c) $f(x,y,z)=x^2+2y^2+3z^2$ $\quad (2,-1,3)$ $\quad \vec{u}=(2,2,2)$
   d) $f(x,y,z)=xe^{yz}$ $\quad (1,1,0)$ $\quad \vec{u}=(-3,0,4)$
   e) $f(x,y,z)=\arctan\frac{x}{y}-\frac{1}{z+1}$ $\quad \left(3,-3,\frac{-1}{2}\right)$ $\quad \vec{u}=(1,1,2)$

3. Find the maximum rate of change of f at the given point and the direction in which it occurs.
   a) $f(x,y)=\dfrac{x}{y^2}$ $\quad (4,2)$
   b) $f(x,y)=e^{3y}\sin 2x$ $\quad \left(\frac{\pi}{4},0\right)$
   c) $f(x,y,z)=\dfrac{x}{x^2+y^2}+\dfrac{z}{x^2+z^2}$ $\quad (3,1,1)$
   d) $f(x,y,z)=\tan(3x+2y+z)$ $\quad (1,1,-5)$

4. Find the directional derivative of $f(x,y)=\sqrt{xy}$ at $P(1,9)$ in the direction of $Q(5,6)$.

5. Find all points at which the direction of fastest change of $f(x,y)=x^2+y^2-4x-2y$ is $(1,1)$.

6. Let $z=3x^2-y^2$. Find all points at which $\|\nabla z\|=6$.

7. Given that $z=3x+y^2$, find $\nabla\|\nabla z\|$ at the point $(5,2)$

8. Marine biologists have determined that a shark detecting the presence of blood will respond by moving continually in the direction in which the concentration of blood increases most rapidly. Suppose that the concentration of blood (in parts per million of water) is given by

$$C(x,y)=e^{\frac{-x^2-2y^2}{10000}}$$

where x and y are horizontal coordinates measured in meters from the blood source. In what direction will the shark move through the point $(3,1)$

9. Suppose that over a certain region of space the electrical potential V is given by $V(x,y,z)=5x^2-3xy+xyz$.
   a) Find the rate of change of the potential at $P(3,4,5)$ in the direction of $\vec{v}=(1,1,-1)$.
   b) In which direction does V change most rapidly at P?
   c) What is the maximum rate of change at P?

10. On a certain mountain, the elevation z in kilometers above a point (x,y) in the xy-plane at sea level is $z=2000-2x^2-4y^2$. The positive x-axis points east and the positive y-axis north. A climber is at the point $(-20,5,1100)$.
    a) If a climber uses a compass reading to walk due west, will she begin to ascend or descend?
    b) If a climber uses a compass reading to walk northeast, will she begin to ascend or descend? At what rate?
    c) In what compass direction should the climber begin walking to travel a level path?

11. Find the equation of the tangent and normal lines to the curve at the given point.
    a) $x^2+x^3y=y^2+11$ at $(2,1)$
    b) $\sqrt{x}+\sqrt{y}=4$ at $(9,1)$

12. Find the equations of the tangent plane and the normal line to each surface at the indicated point.
    a) $x^2+2y^2+3z^2=6$ at $(1,1,1)$
    b) $xyz=24$ at $(2,3,4)$
    c) $y-z=4\arctan(xz)$ at $(1,1+\pi,1)$
    d) $\arctan\frac{y}{x}-\ln xyz=\frac{\pi}{4}$ at $(1,1,1)$
    e) $z=\cos\left(\ln\sqrt{x^2+y^2+1}\right)$ at $(0,0,1)$
    f) $z=xe^{-y}$ at $(1,0,1)$

13. Find all points on the ellipsoid $x^2+2y^2+3z^2=1$ where the tangent plane is parallel to the plane $3x-y+3z=1$.

14. Find the equation of the tangent line to the curve of intersection of the given curves at the given point.
    a) $x^2y^2+2x+z^3=16$ and $3x^2+y^2-2z=9$ at $(2,1,2)$.
    b) $x^2+y^2=4$ and $z=x^2+y^2$ at $\left(\sqrt{2},\sqrt{2},4\right)$
    c) $xyz=1$ and $x^2+2y^2+3z^2=6$ at $(1,1,1)$.

15. Show that the sphere $x^2+y^2+z^2-8x-8y-6z+24=0$ is tangent to the ellipsoid $x^2+3y^2+2z^2=9$ at the point $(2,1,1)$.

16. Show that every plane that is tangent to the cone $x^2+y^2=z^2$ passes through the origin.

17. Show that the equation of the tangent plane to the elliptic paraboloid $\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ at the point (x_0,y_0,z_0) can be written as $\frac{2xx_0}{a^2}+\frac{2yy_0}{b^2}=\frac{z+z_0}{c}$.

18. Find the differential of the function.
    a) $z=x^2\ln y^3$
    b) $z=x\tan(xy)$
    c) $w=\sqrt{x^2+y^2+z^2}$
    d) $u=\frac{r}{s+2t}$

19. Suppose the height of a right circular cone decreases from 20 cm to 19.95 cm, while the radius increases from 4 cm to 4.05 cm. Use differentials to estimate the change in volume.

20. The dimensions of a closed rectangular box are measured as 80cm, 60cm and 50cm respectively, with a possible error of 0.2cm in each direction. Use differentials to estimate the maximum error in the calculated volume of the box.

21. The total resistance R of three resistances R_1, R_2 and R_3, connected in parallel, is given by

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}$$

Suppose that R_1, R_2 and R_3 are measured to be 100 ohms, 200 ohms, and 500 ohms respectively, with a maximum error of 10% in each. Use differentials to approximate the maximum percentage error in the calculated value of R.

22. Find the linearization of f at the given point.
    a) $f(x,y)=xe^y$ at $(3,0)$
    b) $f(x,y)=\arctan(2x+y)$ at $(0,1)$

23. Find the linear approximation of the function $f(x,y)=\sqrt{25-2x^2-3y^2}$ at $(3,1)$ and use it to approximate $f(2.95,0.98)$.

24. Use a linear approximation to estimate the value of $\sqrt{2.98^2+4.05^2}$.

25. Find the Taylor series of the function up to and including quadratic terms.
    a) $f(x,y)=\sin x\cos y$ at $\left(\frac{\pi}{6},\frac{\pi}{3}\right)$
    b) $f(x,y)=e^{y^2-x}$ at $(4,2)$
    c) $f(x,y)=\arctan(xy)$ at $\left(3,\frac{1}{3}\right)$

ANSWERS

1. a) 4 b) $\frac{-5\sqrt{3}}{2}-5$ c) $1-\frac{\sqrt{2}}{2}$

2. a) $\nabla f(x,y)=\left(\frac{2x}{x^2+y^2},\frac{2y}{x^2+y^2}\right)$, $\nabla f(3,4)=\left(\frac{6}{25},\frac{8}{25}\right)$, $D_{\vec{u}}f(3,4)=\frac{-66}{325}$

   b) $\nabla f(x,y)=\left(\frac{x}{\sqrt{x^2-y^2}},\frac{-y}{\sqrt{x^2-y^2}}\right)$, $\nabla f(5,-3)=\left(\frac{5}{4},\frac{3}{4}\right)$, $D_{\vec{u}}f(5,-3)=\frac{11\sqrt{5}}{20}$

   c) $\nabla f(x,y)=(2x,4y,6z)$, $\nabla f(2,-1,3)=(4,-4,18)$, $D_{\vec{u}}f(2,-1,3)=6\sqrt{3}$

   d) $\nabla f(x,y)=\left(e^{yz},xze^{yz},xye^{yz}\right)$, $\nabla f(1,1,0)=(1,0,1)$, $D_{\vec{u}}f(1,1,0)=\frac{1}{5}$

   e) $\nabla f(x,y)=\left(\frac{y}{x^2+y^2},\frac{-x}{x^2+y^2},\frac{1}{(z+1)^2}\right)$, $\nabla f\left(3,-3,\frac{-1}{2}\right)=\left(\frac{-1}{6},\frac{-1}{6},4\right)$, $D_{\vec{u}}f\left(3,-3,\frac{-1}{2}\right)=\frac{23\sqrt{6}}{18}$

3. a) $\left\|\nabla f(4,2)\right\|=\frac{\sqrt{17}}{4}$, $\vec{u}=\left(\frac{\sqrt{17}}{17},\frac{-4\sqrt{17}}{17}\right)$ b) $\left\|\nabla f\left(\frac{\pi}{4},0\right)\right\|=3$, $\vec{u}=(0,1)$

   c) $\left\|\nabla f(3,1,1)\right\|=\frac{\sqrt{74}}{50}$, $\vec{u}=\left(\frac{-7\sqrt{74}}{74},\frac{-3\sqrt{74}}{74},\frac{2\sqrt{74}}{37}\right)$ d) $\left\|\nabla f(1,1,-5)\right\|=\sqrt{14}$, $\vec{u}=\left(\frac{3\sqrt{14}}{14},\frac{\sqrt{14}}{7},\frac{\sqrt{14}}{14}\right)$

4. $D_{\overrightarrow{PQ}}f(1,9)=\frac{11}{10}$ 5. All points on the line $y=x-1$ 6. All points on the ellipse $x^2+\frac{y^2}{9}=1$

7. $\left(0,\frac{8}{5}\right)$ 8. $\left(\frac{-3\sqrt{13}}{13},\frac{-2\sqrt{13}}{13}\right)$ 9. a) $\frac{32\sqrt{3}}{3}$ b) $\left(\frac{19\sqrt{406}}{406},\frac{3\sqrt{406}}{406},\frac{3\sqrt{406}}{203}\right)$ c) $2\sqrt{406}$

10. a) descend b) ascend at $20\sqrt{2}$ km/h c) $\left(\frac{\sqrt{5}}{5},\frac{2\sqrt{5}}{5}\right)$ or $\left(\frac{-\sqrt{5}}{5},\frac{-2\sqrt{5}}{5}\right)$

11. a) $y_T=\frac{-8}{3}x+\frac{19}{3}$ $y_N=\frac{3}{8}x+\frac{1}{4}$ b) $y_T=\frac{-1}{3}x+4$ $y_N=3x-26$

12. a) $\pi_T: x+2y+3z=6$
    $l_N:\vec{l}(t)=(1+t,1+2t,1+3t)$
    b) $\pi_T: 6x+4y+3z=36$
    $l_N:\vec{l}(t)=(2+6t,3+4t,4+3t)$
    c) $\pi_T: 2x-y+3z=4-\pi$
    $l_N:\vec{l}(t)=(1-2t,1+\pi+t,1-3t)$
    d) $\pi_T: 3x+y+2z=6$
    $l_N:\vec{l}(t)=(1+3t,1+t,1+2t)$
    e) $\pi_T: z=1$
    $l_N:\vec{l}(t)=(0,0,1+t)$
    f) $\pi_T: x-y-z=0$
    $l_N:\vec{l}(t)=(1-t,t,1+t)$

13. $\left(\frac{3\sqrt{2}}{5},\frac{-\sqrt{2}}{10},\frac{\sqrt{2}}{5}\right)$ and $\left(\frac{-3\sqrt{2}}{5},\frac{\sqrt{2}}{10},\frac{-\sqrt{2}}{5}\right)$

14. a) $l_T:\vec{l}(t)=(2-10t,1+39t,2-21t)$ b) $l_T:\vec{l}(t)=\left(\sqrt{2}+t,\sqrt{2}-t,4\right)$

    c) $l_T:\vec{l}(t)=(1+t,1-2t,1+t)$

15. $\nabla f(2,1,1)=(-4,-6,-4)=-\nabla g(2,1,1)$ 16. At $P(x_0,y_0,z_0)$, $\pi_T: x_0x+y_0y-z_0z=0$

18. a) $dz=6x\ln y\,dx+\frac{3x^2}{y}dy$ b) $dz=\left(\tan xy+xy\sec^2(xy)\right)dx+x^2\sec^2(xy)\,dy$

    c) $dw=\frac{x}{\sqrt{x^2+y^2+z^2}}dx+\frac{y}{\sqrt{x^2+y^2+z^2}}dy+\frac{z}{\sqrt{x^2+y^2+z^2}}dz$ d) $du=\frac{1}{s+2t}dr-\frac{r}{(s+2t)^2}ds-\frac{2r}{(s+2t)^2}dt$

19. $\frac{12\pi}{5}$ cm^3 20. 2360cm^3 21. 10%

22 a) $L(x,y)=x+3y$ b) $L(x,y)=x+\frac{1}{2}y-\frac{1}{2}+\frac{\pi}{4}$ 23. $\frac{109}{50}$ 24. $\frac{1257}{250}$

25. a) $\sin x\cos y=\frac{1}{4}+\frac{\sqrt{3}}{4}\left(x-\frac{\pi}{6}\right)-\frac{\sqrt{3}}{4}\left(y-\frac{\pi}{3}\right)-\frac{1}{8}\left(x-\frac{\pi}{6}\right)^2-\frac{3}{4}\left(x-\frac{\pi}{6}\right)\left(y-\frac{\pi}{3}\right)-\frac{1}{8}\left(y-\frac{\pi}{3}\right)^2+\cdots$

    b) $e^{y^2-x}=1-(x-4)+4(y-2)+\frac{1}{2}(x-4)^2-4(x-4)(y-2)+9(y-2)^2+\cdots$

    c) $\arctan(xy)=\frac{\pi}{4}+\frac{1}{6}(x-3)+\frac{3}{2}\left(y-\frac{1}{3}\right)-\frac{1}{36}(x-3)^2-\frac{9}{4}\left(y-\frac{1}{3}\right)^2+\cdots$