MATHEMATICS 201-BNK-05
Advanced Calculus
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IV – Arc Length and Curvature

1. Find the arc length of the graph of $\vec{r}(t)$.
   a) $\vec{r}(t)=\left(\cos^3 t,\sin^3 t\right)\quad 0\le t\le \frac{\pi}{2}$
   b) $\vec{r}(t)=\left(e^t,e^{-t},\sqrt{2}t\right)\qquad 0\le t\le 1$
   c) $\vec{r}(t)=\left(12t,8t^{\frac{3}{2}},3t^2\right)\quad 0\le t\le 1$

2. Reparametrize the curve with respect to arc length measured from the point where $t=0$ in the direction of increasing t.
   a) $\vec{r}(t)=\left(\frac{1}{3}t^3,\frac{1}{2}t^2\right)$
   b) $\vec{r}(t)=\left(2t,1-3t,5+4t\right)$
   c) $\vec{r}(t)=\left(\sin e^t,\cos e^t,\sqrt{3}e^t\right)$

3. Find the curvature.
   a) $\vec{r}(t)=\left(t^2,t^3\right)$
   b) $\vec{r}(t)=\left(e^{3t},e^{-t}\right)$
   c) $\vec{r}(t)=\left(t,t,1+t^2\right)$
   d) $\vec{r}(t)=e^t\cos t\mathbf{i}+e^t\sin t\mathbf{j}+e^t\boldsymbol{\kappa}$

4. Find the curvature of he curve at the given point.
   a) $\vec{r}(t)=\left(t,4t^{\frac{3}{2}},-t^2\right)$ at $t=1$
   b) $\vec{r}(t)=\left(\sin t,\cos t,\frac{1}{2}t^2\right)$ at $t=0$

5. Consider the curve $x=x(t)$ and $y=y(t)$ in $\mathbb{R}^2$. Show that the curvature is

$$\kappa(t)=\frac{\left|x'y''-y'x''\right|}{\left(x'^2+y'^2\right)^{\frac{3}{2}}}$$

6. Using the result of number 5, show that the curvature for a plane curve given by $y=f(x)$ is

$$\kappa(x)=\frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$$

7. Use the results from question 6 to find the curvature of the given curves.
   a) $y=\sin x$ at $x=\frac{\pi}{2}$
   b) $y=\frac{1}{x}$ at $x=1$

8. For the function $f(x)=e^x$, at what point does the curve have maximum curvature, and what happens when to the curvature as $x\to\infty$

9. Find the unit tangent, normal and binormal vectors and the given point. Also, find the equation for the tangent line, the normal plane and the osculating plane, at the given point.
   a) $\vec{r}(t)=(2\sin t, 2\cos t, 3t)$ at $(0,2,0)$
   b) $\vec{r}(t)=(e^t, e^t \sin t, e^t \cos t)$ at $(1,0,1)$.
   c) $\vec{r}(t)=(t^2, \frac{2}{3}t^3, t)$ at $t=1$.

10. At what point on the curve $(t^3, 3t, t^4)$ is the normal plane parallel to the plane $6x+6y-8z=1$.

11. Find the equation of the osculating circle for
   a) the curve $y=\frac{1}{2}x^2$ at the origin.
   b) the ellipse $9x^2+4y^2=36$ at the points $(2,0)$ and $(0,3)$

12. Assuming that *s* is an arc length parameter for a smooth vector-valued function $\vec{r}(s)$ in $\mathbb{R}^3$.
   a) Show that $\frac{d\vec{T}}{ds}=\kappa(s)\vec{N}(s)$
   b) Show that $\frac{d\vec{B}}{ds}\perp \vec{B}(s)$
   c) Show that $\frac{d\vec{B}}{ds}\perp \vec{T}(s)$ (*Hint:* differentiate $\vec{B}\bullet\vec{T}$)
   d) Use the results from (b) and (c) to show that $\frac{d\vec{B}}{ds}$ is a scalar multiple of $\vec{N}$.

      The **torsion** $\tau(s)$ is the negative of this scalar, $\frac{d\vec{B}}{ds}=-\tau(s)\vec{N}(s)$.

13. Assuming that *s* is an arc length parameter for a smooth vector-valued function $\vec{r}(s)$ in $\mathbb{R}^3$. Then the following derivatives are known as the **Frenet-Serret formulas**, and are fundamental to the theory of curves.

$$\frac{d\vec{T}}{ds}=\kappa\vec{N}$$

$$\frac{d\vec{N}}{ds}=-k\vec{T}+\tau\vec{B}$$

$$\frac{d\vec{B}}{ds}=-\tau N$$

   Prove the second formula by differentiating $\vec{N}=\vec{B}\times\vec{T}$ with respect to *s*.

14. It can show that for a curve given by $\vec{r}(t)$, the torsion is given by
$$\tau(t)=\frac{\left[\vec{r}'(t)\times r''(t)\right]\bullet r'''(t)}{\left\|\vec{r}'(t)\times r''(t)\right\|^2}$$
Find the torsion for the twisted cubic $\vec{r}(t)=\left(2t,t^2,\frac{1}{3}t^3\right)$.

15. Find the curvature and torsion of the helix $\vec{r}(t)=(a\cos t,a\sin t,bt)$ where a and b are positive constants.

16. Find the velocity, acceleration and speed at an arbitrary time t for the given position vector.
   a) $\vec{r}(t)=\left(e^t,e^{-t}\right)$ b) $\vec{r}(t)=(2t,4t-1,5t+3)$

17. Find the velocity, acceleration and speed at the given time t for the given position vector.
   a) $\vec{r}(t)=\left(\frac{1}{3}t^3,t,\frac{1}{2}t^2\right)$ at $t=1$ b) $\vec{r}(t)=\left(e^t\sin t,e^t\cos t,t\right)$ at $t=\frac{\pi}{2}$

18. Find the position and velocity vectors.
   a) $\vec{a}(t)=\left(\sin t,\cos t,e^t\right)$ $\vec{v}(0)=(0,0,1)$ and $\vec{r}(0)=(-1,0,1)$
   b) $\vec{a}(t)=\frac{1}{(1+t)^2}\mathbf{j}-e^{-t}\mathbf{k}$ $\vec{v}(0)=3\mathbf{i}-\mathbf{j}$ and $\vec{r}(0)=2\mathbf{k}$

19. The position of a particle is given by $\vec{r}(t)=\left(t^2,5t,t^2-16t\right)$. When is the speed minimal?

ANSWERS

1. a) $\frac{3}{2}$ b) $e-\frac{1}{e}$ c) 15

2. a) $\vec{r}(s)=\left(\frac{1}{3}\left((3s+1)^{\frac{2}{3}}-1\right)^{\frac{3}{2}},\frac{1}{2}(3s+1)^{\frac{2}{3}}-\frac{1}{2}\right)$ b) $\vec{r}(s)=\left(\frac{2\sqrt{29}}{29}s,1-\frac{3\sqrt{29}}{29}s,5+\frac{4\sqrt{29}}{29}s\right)$
   c) $\vec{r}(s)=\left(\sin\left(\frac{s}{2}+1\right),\cos\left(\frac{s}{2}+1\right),\frac{\sqrt{3}}{2}s+\sqrt{3}\right)$

3. a) $\frac{6}{t\left(4+9t^2\right)^{\frac{3}{2}}}$ b) $\frac{12e^{2t}}{\left(9e^{6t}+e^{-2t}\right)^{\frac{3}{2}}}$ c) $\frac{1}{\left(1+2t^2\right)^{\frac{3}{2}}}$ d) $\frac{\sqrt{2}}{3}e^{-t}$

4. a) $\frac{7\sqrt{41}}{1681}$ b) $\sqrt{2}$

5. $$\kappa=\frac{\|r'\times r''\|}{\|r'\|^3}=\frac{\left\|\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\x'&y'&0\\x''&y''&0\end{vmatrix}\right\|}{\left(x'^2+y'^2\right)^{\frac{3}{2}}}=\frac{|x'y''-y'x''|}{\left(x'^2+y'^2\right)^{\frac{3}{2}}}$$

6. Let $\vec{r}(x)=(x,f(x))$, then $$\kappa=\frac{|1f''(x)-f'(x)\cdot 0|}{\left(1^2+f'(x)^2\right)^{\frac{3}{2}}}=\frac{\left|\frac{d^2y}{dx^2}\right|}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}$$

7. a) 1 b) $\frac{\sqrt{2}}{2}$

8. Max at $\left(\frac{-1}{2}\ln 2, \frac{\sqrt{2}}{2}\right)$, and $\lim_{x\to\infty}\kappa(x) = \lim_{x\to\infty}\frac{e^x}{\left(1+e^{2x}\right)^{\frac{3}{2}}} = 0$

9. a) $\vec{T}(0) = \frac{\sqrt{13}}{13}(2,0,3)$ tangent line: $(x,y,z) = (0,2,0) + t(2,0,3)$
$\vec{N}(0) = (0,-1,0)$ normal plane: $2x + 3z = 0$
$\vec{B}(0) = \frac{\sqrt{13}}{13}(3,0,-2)$ osculating plane: $3x - 2z = 0$

b) $\vec{T}(0) = \frac{\sqrt{3}}{3}(1,1,1)$ tangent line: $(x,y,z) = (1,0,1) + t(1,1,1)$
$\vec{N}(0) = \frac{\sqrt{2}}{2}(0,1,-1)$ normal plane: $x + y + z = 2$
$\vec{B}(0) = \frac{\sqrt{6}}{6}(-2,1,1)$ osculating plane: $2x - y - z = 1$

c) $\vec{T}(0) = \frac{1}{3}(2,2,1)$ tangent line: $(x,y,z) = \left(1,\frac{2}{3},1\right) + t(2,2,1)$
$\vec{N}(0) = \frac{1}{3}(-1,2,-2)$ normal plane: $2x + 2y + z = \frac{13}{3}$
$\vec{B}(0) = \frac{1}{3}(-2,1,2)$ osculating plane: $2x - y - 2z = \frac{-2}{3}$

10. $(-1,-3,1)$

11. a) $x^2 + (y-1)^2 = 1$ b) $\left(x+\frac{5}{2}\right)^2 + y^2 = \frac{81}{4}$, $x^2 + \left(y-\frac{5}{3}\right)^2 = \frac{16}{9}$

12. a) $\vec{T}(s) = \vec{r}\,'(s)$
$\vec{T}\,'(s) = \vec{r}\,''(s) = \left\|\vec{r}\,''(s)\right\|\frac{\vec{r}\,''(s)}{\left\|\vec{r}\,''(s)\right\|} = k(s)\vec{N}(s)$

b) $\vec{B}(s)\bullet\vec{B}(s) = 1$
$\frac{d\vec{B}}{ds}\bullet\vec{B}(s) + \vec{B}(s)\bullet\frac{d\vec{B}}{ds} = 0$
$\frac{d\vec{B}}{ds}\bullet\vec{B}(s) = 0$

c) $\vec{B}\bullet\vec{T} = 0$ since $\vec{B}\perp\vec{T}$
$\frac{d\vec{B}}{ds}\bullet\vec{T} + B\bullet\frac{d\vec{T}}{ds} = 0$
$\frac{d\vec{B}}{ds}\bullet\vec{T} + B\bullet\left(k\vec{N}\right) = 0$ since $\vec{B}\perp\vec{N}$
$\frac{d\vec{B}}{ds}\bullet\vec{T} = 0$

d) by (b) and (c), $\frac{d\vec{B}}{ds}$ is parallel to $\vec{B}\times\vec{T} = \vec{N}$

13. $\vec{N} = \vec{B}\times\vec{T}$
$\frac{d\vec{N}}{ds} = \frac{d\vec{B}}{ds}\times\vec{T} + \vec{B}\times\frac{d\vec{T}}{ds}$
$= -\tau\vec{N}\times\vec{T} + \vec{B}\times k\vec{N}$
$= -k\vec{T} + \tau\vec{B}$

14. $\frac{2}{\left(t^2+2\right)^2}$

15. $\kappa = \frac{a}{a^2+b^2}$ $\tau = \frac{b}{a^2+b^2}$

16. a) $\vec{v}(t) = \left(e^t, -e^{-t}\right)$ $\vec{a}(t) = \left(e^t, e^{-t}\right)$ $\left\|\vec{v}(t)\right\| = \sqrt{e^{2t} + e^{-2t}}$
b) $\vec{v}(t) = (2,4,5)$ $\vec{a}(t) = (0,0,0)$ $\left\|\vec{v}(t)\right\| = 3\sqrt{5}$

17. a) $\vec{v}(t) = (1,1,1)$ $\vec{a}(t) = (2,0,1)$ $\left\|\vec{v}(t)\right\| = \sqrt{3}$
b) $\vec{v}(t) = \left(e^{\frac{\pi}{2}}, -e^{\frac{\pi}{2}}, 1\right)$ $\vec{a}(t) = \left(0, -2e^{\frac{\pi}{2}}, 0\right)$ $\left\|\vec{v}(t)\right\| = \sqrt{1+2e^{\pi}}$

18. a) $\vec{v}(t) = \left(1-\cos t, \sin t, e^t\right)$
$\vec{r}(t) = \left(t - \sin t - 1, 1 - \cos t, e^t\right)$

b) $\vec{v}(t) = \left(3, \frac{-1}{1+t}, e^{-t} - 1\right)$
$\vec{r}(t) = \left(3t, -\ln(t+1), -e^{-t} - t + 3\right)$

19. $t = 4$