



MATHEMATICS 201-BNK-05

Advanced Calculus

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II – Continuity

SOLUTIONS

1. Use the formal definition for continuity to prove that the following functions are continuous at the given point.

a) $f(x) = 3x + 5$ at $x = 2$

Let $\varepsilon > 0$ be given.

$$\begin{aligned} |f(x) - L| &= |3x + 5 - 11| & \text{Hyp: } |x - 2| < \delta \\ &= 3|x - 2| \\ &< 3\delta = \varepsilon \end{aligned}$$

If $\delta = \frac{\varepsilon}{3}$ then $|x - 2| < \delta \Rightarrow |f(x) - f(2)| < \varepsilon$, ergo f is continuous at $x = 2$

b) $f(x) = -4x + 1$ at $x = -5$

Let $\varepsilon > 0$ be given.

$$\begin{aligned} |f(x) - f(-5)| &= |-4x + 1 - 21| & \text{Hyp: } |x + 5| < \delta \\ &= 4|x + 5| \\ &< 4\delta = \varepsilon \end{aligned}$$

If $\delta = \frac{\varepsilon}{4}$ then $|x + 5| < \delta \Rightarrow |f(x) - f(-5)| < \varepsilon$, ergo f is continuous at $x = -5$

c) $f(x) = 2x^2 - 3$ at $x = 1$

Let $\varepsilon > 0$ be given.

$$\begin{aligned} |f(x) - f(1)| &= |2x^2 - 3 + 1| & \text{Hyp: } |x - 1| < \delta \\ &= 2|x - 1||x + 1| & \text{Let } \delta \leq 1 \\ &< 6\delta = \varepsilon & \begin{aligned} -1 < x - 1 < 1 \\ 1 < x + 1 < 3 \end{aligned} \end{aligned}$$

If $\delta = \min\{1, \frac{\varepsilon}{6}\}$ then $|x - 1| < \delta \Rightarrow |f(x) - f(1)| < \varepsilon$, ergo f is cont. at $x = 1$.

d) $f(x) = x^2 - 5x + 2$ at $x = -2$

Let $\varepsilon > 0$ be given.

$$\begin{aligned} |f(x) - f(-2)| &= |x^2 - 5x + 2 - 16| & \text{Hyp: } |x + 2| < \delta \\ &= |x + 2||x - 7| & \text{Let } \delta \leq 1 \\ &< 10\delta = \varepsilon & \begin{aligned} -1 < x + 2 < 1 \\ -10 < x - 7 < -8 \\ 8 < |x - 7| < 10 \end{aligned} \end{aligned}$$

If $\delta = \min\{1, \frac{\varepsilon}{10}\}$ then $|x + 2| < \delta \Rightarrow |f(x) - f(-2)| < \varepsilon$, ergo f is cont. at $x = -2$.

e) $f(x) = \frac{3}{x}$ at $x = -6$

Let $\varepsilon > 0$ be given.

$$\begin{aligned} |f(x) - f(-6)| &= \left| \frac{3}{x} + \frac{1}{2} \right| \\ &= \frac{|x+6|}{2|x|} \\ &< \frac{\delta}{10} = \varepsilon \end{aligned}$$

Hyp: $|x+6| < \delta$

Let $\delta \leq 1$ $-1 < x+6 < 1$

$$-7 < x < -5$$

$$\frac{1}{7} < \frac{1}{|x|} < \frac{1}{5}$$

If $\delta = \min\{1, 10\varepsilon\}$ then $|x+6| < \delta \Rightarrow |f(x) - f(-6)| < \varepsilon$, ergo f is continuous at $x = -6$.

f) $f(x) = \frac{5}{x^2}$ at $x = -1$

Let $\varepsilon > 0$ be given.

$$\begin{aligned} |f(x) - f(-1)| &= \left| \frac{5}{x^2} - 5 \right| \\ &= \frac{5|x+1||x-1|}{x^2} \\ &< 5 \cdot 4 \cdot \frac{\delta}{2} = \varepsilon \end{aligned}$$

Hyp: $|x+1| < \delta$

Let $\delta \leq \frac{1}{2}$ $-\frac{1}{2} < x+1 < \frac{1}{2}$

$$\frac{1}{2} < |x| < \frac{3}{2}$$

$$\frac{4}{9} < \frac{1}{x^2} < 4$$

$$\frac{3}{2} < x+1 < \frac{5}{2}$$

If $\delta = \min\{\frac{1}{2}, \frac{\varepsilon}{50}\}$ then $|x+1| < \delta \Rightarrow |f(x) - f(-1)| < \varepsilon$, ergo f is continuous at $x = -1$.

g) $f(x) = \frac{4}{2x-5}$ at $x = 2$

Let $\varepsilon > 0$ be given.

$$\begin{aligned} |f(x) - f(2)| &= \left| \frac{4}{2x-5} + 4 \right| \\ &= \frac{8|x-2|}{|2x-5|} \\ &< 8 \cdot 2\delta = \varepsilon \end{aligned}$$

Hyp: $|x-2| < \delta$

Let $\delta \leq \frac{1}{4}$ $-\frac{1}{4} < x-2 < \frac{1}{4}$

$$\frac{7}{4} < x < \frac{9}{4}$$

$$-\frac{3}{2} < 2x-5 < -\frac{1}{2}$$

$$\frac{1}{2} < |2x-5| < \frac{3}{2}$$

$$\frac{2}{3} < \frac{1}{|2x-5|} < 2$$

If $\delta = \min\{\frac{1}{4}, \frac{\varepsilon}{16}\}$ then $|x-2| < \delta \Rightarrow |f(x) - f(2)| < \varepsilon$, ergo f is continuous at $x = 2$.

h) $f(x) = 2x^3 - 1$ at $x = -1$

Let $\varepsilon > 0$ be given.

$$\begin{aligned} |f(x) - f(-1)| &= |2x^3 - 1 + 3| && \text{Hyp: } |x+1| < \delta \\ &= 2|x+1||x^2 - x + 1| && \text{Let } \delta \leq \frac{1}{2} \quad -\frac{1}{2} < x+1 < \frac{1}{2} \\ &\leq 2|x+1|(x^2 + |x| + 1) && \frac{-3}{2} < x < \frac{-1}{2} \\ &< 2 \cdot \frac{19}{4} \delta = \varepsilon && \frac{1}{2} < |x| < \frac{3}{2} \\ &&& \frac{1}{4} < |x^2| < \frac{9}{4} \\ &&& \frac{7}{4} < x^2 + |x| + 1 < \frac{19}{4} \end{aligned}$$

If $\delta = \min\{\frac{1}{2}, \frac{2\varepsilon}{19}\}$ then $|x+1| < \delta \Rightarrow |f(x) - f(-1)| < \varepsilon$, ergo f is continuous at $x = -1$.

2. Show that the following functions are discontinuous at the given point.

a) $f(x) = \begin{cases} 2 & \text{if } x \leq 1 \\ 3 & \text{if } x > 1 \end{cases} \quad x = 1$

Let $\varepsilon = \frac{1}{2}$. Then for all values of δ , there will be values of x on the interval $(1 - \delta, 1 + \delta)$ such that $x > 1$. For these values, $|f(x) - f(1)| = |3 - 2| = 1 > \varepsilon = \frac{1}{2}$.

Hence f is not continuous at $x = 1$.

b) $f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ x & \text{if } x > 2 \end{cases} \quad x = 2$

Let $\varepsilon = 1$. Then for all values of δ , there will be values of x on the interval $(2 - \delta, 2 + \delta)$ such that $2 < x < 3$. For these values,

$$\begin{aligned} 2 < x < 3 \\ -2 < x - 4 < -1 \\ 1 < |x - 4| < 2 \end{aligned}$$

thus $|f(x) - f(2)| = |x - 4| > \varepsilon = 1$. Hence f is not continuous at $x = 2$.

c) $f(x) = \begin{cases} \frac{1}{x} & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases} \quad x = 0$

Let $\varepsilon = 1$. Then for all values of δ , there will be values of x on the interval $(-\delta, \delta)$ such that $-1 < x < 0$. For these values,

$$\begin{aligned} -1 < x < 0 \\ 0 < |x| < 1 \\ 1 < \frac{1}{|x|} \end{aligned}$$

thus $|f(x) - f(0)| = |\frac{1}{x} - 0| > \varepsilon = 1$. Hence f is not continuous at $x = 0$.

$$d) f(x) = \begin{cases} 2x+1 & \text{if } x \leq 3 \\ 3-x & \text{if } x > 3 \end{cases} \quad x = 3$$

Let $\varepsilon = 1$. Then for all values of δ , there will be values of x on the interval $(3-\delta, 3+\delta)$ such that $3 < x < 4$. For these values,

$$3 < x < 4$$

$$7 < x+4 < 11$$

thus $|f(x) - f(3)| = |3-x-7| = |x+4| > 7 > \varepsilon = 1$. Hence f is not continuous at $x = 3$.

$$e) f(x) = \begin{cases} x & \text{if } x \neq -4 \\ 5 & \text{if } x = -4 \end{cases} \quad x = -4$$

Let $\varepsilon = 1$. Then for all values of δ , there will be values of x on the interval $(-4-\delta, -4+\delta)$ such that $-5 < x < -4$. For these values,

$$-5 < x < -4$$

$$-10 < x-5 < -9$$

$$9 < |x-5| < 10$$

thus $|f(x) - f(-4)| = |x-5| > 9 > \varepsilon = 1$. Hence f is not continuous at $x = -4$.

$$f) f(x) = \begin{cases} x-4 & \text{if } x < -4 \\ 4-x & \text{if } x \geq -4 \end{cases} \quad x = -4$$

Let $\varepsilon = 1$. Then for all values of δ , there will be values of x on the interval $(-4-\delta, -4+\delta)$ such that $-5 < x < -4$. For these values,

$$-5 < x < -4$$

$$-17 < x-12 < -16$$

thus $|f(x) - f(-4)| = |x-4-8| = |x-12| > 16 > \varepsilon = 1$. Hence f is not continuous at $x = -4$.

3. Show that the following functions are continuous everywhere.

$$a) f(x) = 5x + 2$$

Let a be any number and let $\varepsilon > 0$ be given.

$$\begin{aligned} |f(x) - f(a)| &= |5x + 2 - 5a - 2| && \text{Hyp: } |x - a| < \delta \\ &= 5|x - a| \\ &< 5\delta = \varepsilon \end{aligned}$$

If $\delta = \frac{\varepsilon}{5}$ then $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$, ergo f is continuous at $x = a$.

b) $f(x) = x^2 - 4$

Let a be any number and let $\varepsilon > 0$ be given.

$$\begin{aligned} |f(x) - f(a)| &= |x^2 - 4 - a^2 + 4| && \text{Hyp: } |x - a| < \delta \\ &= |x + a||x - a| && \text{Let } \delta \leq 1 \\ &< (1 + 2|a|)\delta = \varepsilon && \begin{aligned} -1 < x - a < 1 \\ -1 + 2a < x + a < 1 + 2a < 1 + 2|a| \end{aligned} \end{aligned}$$

If $\delta = \min\left\{1, \frac{\varepsilon}{1+2|a|}\right\}$ then $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$, ergo f is continuous at $x = a$.

c) $f(x) = \frac{1}{x^2}$

Let a be any number such that $a \neq 0$ and let $\varepsilon > 0$ be given.

$$\begin{aligned} |f(x) - f(a)| &= \left|\frac{1}{x^2} - \frac{1}{a^2}\right| && \text{Hyp: } |x - a| < \delta \\ &= \frac{|x+a||x-a|}{x^2 a^2} && \text{Let } \delta \leq \frac{|a|}{2} \quad \begin{aligned} -\frac{|a|}{2} < x - a < \frac{|a|}{2} \\ a - \frac{|a|}{2} < x < a + \frac{|a|}{2} \\ \frac{|a|}{2} < |x| < \frac{3|a|}{2} \\ \frac{4}{9a^2} < \frac{1}{x^2} < \frac{4}{a^2} \end{aligned} \\ &< \frac{5|a|}{2} \frac{1}{4a^2 a^2} \delta = \varepsilon && |x+a| < |x| + |a| < \frac{5|a|}{2} \end{aligned}$$

If $\delta = \min\left\{\frac{|a|}{2}, \frac{8a^4\varepsilon}{5|a|}\right\}$ then $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$, ergo f is continuous at $x = a$.

d) $f(x) = \sqrt{x}$ for $x > 0$

Let a be any number such that $a > 0$ and let $\varepsilon > 0$ be given.

$$\begin{aligned} |f(x) - f(a)| &= |\sqrt{x} - \sqrt{a}| && \text{Hyp: } |x - a| < \delta \\ &= \frac{|x-a|}{\sqrt{x} + \sqrt{a}} && \text{Let } \delta \leq \frac{a}{2} \\ &< \frac{\sqrt{2}}{\sqrt{a} + \sqrt{2a}} \delta = \varepsilon && \begin{aligned} -\frac{a}{2} < x - a < \frac{a}{2} \\ \frac{a}{2} < x < \frac{3a}{2} \\ \sqrt{\frac{a}{2}} + \sqrt{a} < \sqrt{x} + \sqrt{a} < \sqrt{\frac{3a}{2}} + \sqrt{a} \\ \frac{1}{\sqrt{\frac{3a}{2}} + \sqrt{a}} < \frac{1}{\sqrt{x} + \sqrt{a}} < \frac{1}{\sqrt{\frac{a}{2}} + \sqrt{a}} \\ \frac{\sqrt{2}}{\sqrt{3a} + \sqrt{2a}} < \frac{1}{\sqrt{x} + \sqrt{a}} < \frac{\sqrt{2}}{\sqrt{a} + \sqrt{2a}} \end{aligned} \end{aligned}$$

If $\delta = \min\left\{\frac{|a|}{2}, \frac{\sqrt{2}}{\sqrt{a} + \sqrt{2a}}\varepsilon\right\}$ then $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$, ergo f is continuous at $x = a$.

4. Prove the following properties. (These proofs resemble the ones for limits)

a) If $f(x)$ is continuous at $x = a$, then $cf(x)$ is continuous at $x = a$.

Proof:

Let $\varepsilon > 0$ be given.

Since f is continuous then $\forall \varepsilon_1 > 0$,

$$\begin{aligned} |cf(x) - cf(a)| &= c|f(x) - f(a)| && \exists \delta > 0 \text{ such that} \\ &< c\varepsilon_1 && |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon_1 \end{aligned}$$

If we let $\varepsilon_1 = \frac{\varepsilon}{c}$, then $|x - a| < \delta \Rightarrow |cf(x) - cf(a)| < \varepsilon$.

Thus $cf(x)$ is continuous at $x = a$

b) If $f(x)$ and $g(x)$ are continuous at $x = a$, then $f(x) + g(x)$ is continuous at $x = a$.

Proof:

Since f is continuous then $\forall \varepsilon_1 > 0$, $\exists \delta_1 > 0$ such that

$$|x - a| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon_1$$

Since g is continuous then $\forall \varepsilon_2 > 0$, $\exists \delta_2 > 0$ such that

$$|x - a| < \delta_2 \Rightarrow |g(x) - L| < \varepsilon_2$$

Let $\varepsilon > 0$ be given.

$$\begin{aligned} |f(x) + g(x) - f(a) - g(a)| &= |f(x) - f(a) + g(x) - g(a)| \\ &\leq |f(x) - f(a)| + |g(x) - g(a)| \\ &< \varepsilon_1 + \varepsilon_2 \end{aligned}$$

If we let $\varepsilon_1 = \frac{\varepsilon}{2}$ and $\varepsilon_2 = \frac{\varepsilon}{2}$, then and if $\delta = \min\{\delta_1, \delta_2\}$ then

$$|x - a| < \delta \Rightarrow |f(x) + g(x) - f(a) - g(a)| < \varepsilon$$

c) If $f(x)$ and $g(x)$ are continuous at $x = a$, then $f(x)g(x)$ is continuous at $x = a$.

Proof:

Since f is continuous then $\forall \varepsilon_1 > 0$, $\exists \delta_1 > 0$ such that

$$|x - a| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon_1$$

and $\exists N > 0$ and $\delta_2 > 0$ such that $0 < |x - a| < \delta_2 \Rightarrow |f(x)| < N$

Since g is continuous then $\forall \varepsilon_2 > 0$, $\exists \delta_3 > 0$ such that

$$|x - a| < \delta_3 \Rightarrow |g(x) - L| < \varepsilon_2$$

Let $\varepsilon > 0$ be given.

$$\begin{aligned} |f(x)g(x) - f(a)g(a)| &= |f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)| \\ &\leq |f(x)g(x) - f(x)g(a)| + |f(x)g(a) - f(a)g(a)| \\ &= |f(x)||g(x) - g(a)| + |g(a)||f(x) - f(a)| \\ &< N\varepsilon_2 + |g(a)|\varepsilon_1 \end{aligned}$$

If we let $\varepsilon_1 = \begin{cases} \frac{\varepsilon}{2g(a)} & g(a) \neq 0 \\ \varepsilon & g(a) = 0 \end{cases}$ and $\varepsilon_2 = \begin{cases} \frac{\varepsilon}{2N} & g(a) \neq 0 \\ \frac{\varepsilon}{N} & g(a) = 0 \end{cases}$,

and if $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ then $|x - a| < \delta \Rightarrow |f(x)g(x) - f(a)g(a)| < \varepsilon$