MATHEMATICS 201-BNK-05
Advanced Calculus
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I – Formal Limits

SOLUTIONS

1. Use the definition of a limit to prove the following statements.

a) $\lim_{x\to 3}(x-7)=-4$

Let $\varepsilon>0$ be given.

$$\begin{aligned}|f(x)-L| &= |x-7-(-4)| \\ &= |x-3| \\ &< \delta=\varepsilon\end{aligned}$$

Hyp: $0<|x-3|<\delta$

If $\delta=\varepsilon$ then $0<|x-3|<\delta \Rightarrow |f(x)-(-4)|<\varepsilon$, ergo $\lim_{x\to 3}(x-7)=-4$.

b) $\lim_{x\to 2}(2x+3)=7$

Let $\varepsilon>0$ be given.

$$\begin{aligned}|f(x)-L| &= |2x+3-7| \\ &= 2|x-2| \\ &< 2\delta=\varepsilon\end{aligned}$$

Hyp: $0<|x-2|<\delta$

If $\delta=\frac{\varepsilon}{2}$ then $0<|x-2|<\delta \Rightarrow |f(x)-7|<\varepsilon$, ergo $\lim_{x\to 2}(2x+3)=7$.

c) $\lim_{x\to 0}x^2=0$

Let $\varepsilon>0$ be given.

$$\begin{aligned}|f(x)-L| &= |x^2-0| \\ &= |x|^2 \\ &< \delta^2=\varepsilon\end{aligned}$$

Hyp: $0<|x|<\delta$

If $\delta=\sqrt{\varepsilon}$ then $0<|x|<\delta \Rightarrow |f(x)-0|<\varepsilon$, ergo $\lim_{x\to 0}x^2=0$

d) $\lim_{x\to 3}(x^2-x)=6$

Let $\varepsilon>0$ be given.

$$\begin{aligned}|f(x)-L| &= |x^2-x-6| \\ &= |x+2||x-3| \\ &< 6\delta=\varepsilon\end{aligned}$$

Hyp: $0<|x-3|<\delta$

Let $\delta\le 1$, then $-1<x-3<1$

$4<x+5<6$

If $\delta=\min\{1,\frac{\varepsilon}{6}\}$ then $0<|x-3|<\delta \Rightarrow |f(x)-6|<\varepsilon$, ergo $\lim_{x\to 3}(x^2-x)=6$

e) $\lim_{x\to 2}\frac{1}{x}=\frac{1}{2}$

Let $\varepsilon>0$ be given.

$|f(x)-L|=\left|\frac{1}{x}-\frac{1}{2}\right|$

$=\frac{|2-x|}{2|x|}$

$<\frac{\delta}{2}=\varepsilon$

Hyp: $0<|x-2|<\delta$

Let $\delta\le 1$, then $-1<x-2<1$

$1<x<3$

$\frac{1}{3}<\frac{1}{x}<1$

If $\delta=\min\{1,2\varepsilon\}$ then $0<|x-2|<\delta\Rightarrow\left|f(x)-\frac{1}{2}\right|<\varepsilon$, ergo $\lim_{x\to 2}\frac{1}{x}=\frac{1}{2}$

f) $\lim_{x\to 2}\frac{3}{2x-5}=-3$

Let $\varepsilon>0$ be given.

$|f(x)-L|=\left|\frac{3}{2x-5}+3\right|$

$=\frac{6|x-2|}{|2x-5|}$

$<12\delta=\varepsilon$

Hyp: $0<|x-2|<\delta$

Let $\delta\le\frac{1}{4}$, then $-\frac{1}{4}<x-2<\frac{1}{4}$

$\frac{7}{4}<x<\frac{9}{4}$

$\frac{7}{2}<2x<\frac{9}{2}$

$\frac{-3}{2}<2x-5<\frac{-1}{2}$

$\frac{2}{3}<\frac{1}{|2x-5|}<2$

If $\delta=\min\{\frac{1}{4},\frac{\varepsilon}{12}\}$ then $0<|x-2|<\delta\Rightarrow|f(x)+3|<\varepsilon$, ergo $\lim_{x\to 2}\frac{3}{2x-5}=-3$

g) $\lim_{x\to 2}x^3=8$

Let $\varepsilon>0$ be given.

$|f(x)-L|=|x^3-8|$

$=|x-2||x^2+2x+4|$

$\le|x-2|\left(x^2+2|x|+4\right)$

$<19\delta=\varepsilon$

Hyp: $0<|x-2|<\delta$

Let $\delta\le 1$, then $-1<x-2<1$

$1<x<3$

$1<x^2<9$

If $\delta=\min\{1,\frac{\varepsilon}{19}\}$ then $0<|x-2|<\delta\Rightarrow|f(x)-8|<\varepsilon$, ergo $\lim_{x\to 2}x^3=8$

h) $\lim_{x\to 9}\sqrt{x}=3$

Let $\varepsilon>0$ be given.

$|f(x)-L|=|\sqrt{x}-3|\cdot\frac{|\sqrt{x}+3|}{|\sqrt{x}+3|}$

$=\frac{|x-9|}{|\sqrt{x}+3|}$

$<\frac{\delta}{\sqrt{8}+3}=\varepsilon$

Hyp: $0<|x-9|<\delta$

Let $\delta\le 1$, then $-1<x-9<1$

$8<x<10$

$\sqrt{8}+3<\sqrt{x}+3<\sqrt{10}+3$

$\frac{1}{\sqrt{10}+3}<\frac{1}{\sqrt{x}+3}<\frac{1}{\sqrt{8}+3}$

If $\delta=\min\left\{1,\varepsilon\left(\sqrt{8}+3\right)\right\}$ then $0<|x-9|<\delta\Rightarrow|f(x)-3|<\varepsilon$, ergo $\lim_{x\to 9}\sqrt{x}=3$

i) $\lim\limits_{x\to 2}\dfrac{x}{x^2-3}=2$

Let $\varepsilon>0$ be given.

$$\begin{aligned}|f(x)-L| &= \left|\tfrac{x}{x^2-3}-2\right| \\ &= \frac{|2x+3||x-2|}{|x^2-3|} \\ &< \tfrac{15}{2}16\delta=\varepsilon\end{aligned}$$

Hyp: $0<|x-2|<\delta$

Let $\delta\le\frac{1}{4}$, then
$$\begin{aligned}-\tfrac{1}{4}&<x-2<\tfrac{1}{4}\\ \tfrac{7}{4}&<x<\tfrac{9}{4}\\ \tfrac{13}{2}&<2x+3<\tfrac{15}{2}\\ \tfrac{49}{16}&<x^2<\tfrac{81}{16}\\ \tfrac{1}{16}&<x^2-3<\tfrac{33}{16}\\ \tfrac{16}{33}&<\tfrac{1}{|x^2-3|}<16\end{aligned}$$

If $\delta=\min\left\{\frac{1}{4},\frac{\varepsilon}{120}\right\}$ then $0<|x-2|<\delta\Rightarrow|f(x)-2|<\varepsilon$, ergo $\lim\limits_{x\to 2}\dfrac{x}{x^2-3}=2$

j) $\lim\limits_{x\to -2}\dfrac{2}{\sqrt{x^2-6x}}=\dfrac{1}{2}$

Let $\varepsilon>0$ be given.

$$\begin{aligned}|f(x)-L| &= \left|\tfrac{2}{\sqrt{x^2-6x}}-\tfrac{1}{2}\right| \\ &= \left|\tfrac{4-\sqrt{x^2-6x}}{2\sqrt{x^2-6x}}\cdot\tfrac{4+\sqrt{x^2-6x}}{4+\sqrt{x^2-6x}}\right| \\ &= \tfrac{|16-x^2+6x|}{2|\sqrt{x^2-6x}||4+\sqrt{x^2-6x}|} \\ &= \tfrac{|-x+8||x+2|}{2|\sqrt{x^2-6x}||4+\sqrt{x^2-6x}|} \\ &< 11\tfrac{1}{2}\tfrac{1}{\sqrt{7}}\tfrac{1}{4+\sqrt{7}}\delta=\varepsilon\end{aligned}$$

Hyp: $0<|x+2|<\delta$

Let $\delta\le 1$, then
$$\begin{aligned}-1&<x+2<1\\ -11&<x-8<-9\\ -3&<x<-1\\ 1&<x^2<9\\ 6&<-6x<18\\ 7&<x^2-6x<27\\ 4+\sqrt{7}&<4+\sqrt{x^2-6x}<4+\sqrt{27}\end{aligned}$$

If $\delta=\min\left\{1,\frac{14+8\sqrt{7}}{11}\varepsilon\right\}$ then $0<|x+2|<\delta\Rightarrow\left|f(x)-\frac{1}{2}\right|<\varepsilon$, ergo $\lim\limits_{x\to -2}\dfrac{2}{\sqrt{x^2-6x}}=\dfrac{1}{2}$

k) $\lim\limits_{x\to 3}\dfrac{x^2-x-6}{x-3}=5$

Let $\varepsilon>0$ be given.

$$\begin{aligned}|f(x)-L| &= \left|\tfrac{x^2-x-6}{x-3}-5\right| \\ &= |x-3| \\ &< \delta=\varepsilon\end{aligned}$$

Hyp: $0<|x-3|<\delta$

If $\delta=\varepsilon$ then $0<|x-3|<\delta\Rightarrow|f(x)-5|<\varepsilon$, ergo $\lim\limits_{x\to 3}\dfrac{x^2-x-6}{x-3}=5$

m) $\lim_{x\to 3}\left(2x^2-5x+1\right)=4$

Let $\varepsilon>0$ be given.

$$\begin{aligned}\left|f(x)-L\right| &= \left|2x^2-5x+1-4\right| \\ &= \left|2x+1\right|\left|x-3\right| \\ &< 9\delta=\varepsilon\end{aligned}$$

Hyp: $0<\left|x-3\right|<\delta$

Let $\delta\le 1$, then $-1<x-3<1$

$$2<x<4$$
$$5<2x+1<9$$

If $\delta=\min\left\{1,\frac{\varepsilon}{9}\right\}$ then $0<\left|x-3\right|<\delta\Rightarrow\left|f(x)-4\right|<\varepsilon$, ergo $\lim_{x\to 3}\left(2x^2-5x+1\right)=4$

n) $\lim_{x\to 2}\left|5x-3\right|=7$

Let $\varepsilon>0$ be given.

$$\begin{aligned}\left|f(x)-L\right| &= \left|\left|5x-3\right|-7\right| \\ &= 5\left|x-2\right| \\ &< 5\delta=\varepsilon\end{aligned}$$

Hyp: $0<\left|x-2\right|<\delta$

Let $\delta\le 1$, then $-1<x-2<1$

$$1<x<3$$
$$2<5x-3<12$$

Thus $\left|5x-3\right|=5x-3$

If $\delta=\min\left\{1,\frac{\varepsilon}{5}\right\}$ then $0<\left|x-2\right|<\delta\Rightarrow\left|f(x)-7\right|<\varepsilon$, ergo $\lim_{x\to 2}\left|5x-3\right|=7$

o) $\lim_{x\to 3}\sqrt{5x+1}=4$

Let $\varepsilon>0$ be given.

$$\begin{aligned}\left|f(x)-L\right| &= \left|\sqrt{5x+1}-4\right|\cdot\frac{\left|\sqrt{5x+1}+4\right|}{\left|\sqrt{5x+1}+4\right|} \\ &= \frac{5\left|x-3\right|}{\left|\sqrt{5x+1}+4\right|} \\ &< \frac{5\delta}{\sqrt{11}+4}=\varepsilon\end{aligned}$$

Hyp: $0<\left|x-3\right|<\delta$

Let $\delta\le 1$, then

$$-1<x-3<1$$
$$2<x<4$$
$$\sqrt{11}+4<\sqrt{5x+1}+4<\sqrt{21}+4$$
$$\frac{1}{\sqrt{21}+4}<\frac{1}{\sqrt{5x+1}+4}<\frac{1}{\sqrt{11}+4}$$

If $\delta=\min\left\{1,\frac{\varepsilon\left(\sqrt{11}+4\right)}{5}\right\}$ then $0<\left|x-3\right|<\delta\Rightarrow\left|f(x)-4\right|<\varepsilon$, ergo $\lim_{x\to 3}\sqrt{5x+1}=4$

2. Use the definition of a limit to show that the following statements are false.

a) $\lim_{x\to 2}\left(2x-1\right)=5$

Let $\varepsilon=1$. Then for all values of δ, there will be values of x on the interval $\left(2-\delta,2+\delta\right)$ such that $0<x<2$. For these values,

$$0<x<2$$
$$-6<2x-6<-2$$
$$2<\left|2x-6\right|<6$$

thus $\left|f(x)-5\right|=\left|2x-6\right|>2>\varepsilon=1$. Hence $\lim_{x\to 2}\left(2x-1\right)\ne 5$

b) $\lim_{x\to 5} x^2 = 20$

Let $\varepsilon = 1$. Then for all values of δ, there will be values of x on the interval $(5-\delta, 5+\delta)$ such that $x > 5$. For these values,

$$x > 5$$
$$x^2 > 25$$
$$x^2 - 20 > 5$$

thus $|f(x) - 20| = |x^2 - 20| > 5 > \varepsilon = 1$. Hence $\lim_{x\to 5} x^2 \neq 20$

c) $\lim_{x\to 1} \frac{4}{x+1} = 1$

Let $\varepsilon = 1$. Then for all values of δ, there will be values of x on the interval $(1-\delta, 1+\delta)$ such that $0 < x < 1$. For these values,

$$0 < x < 1$$
$$1 < x+1 < 2$$
$$\tfrac{1}{2} < \tfrac{1}{x+1} < 1$$
$$1 < \tfrac{4}{x+1} - 1 < 3$$

thus $|f(x) - 1| = \left|\tfrac{4}{x+1} - 1\right| > \varepsilon = 1$. Hence $\lim_{x\to 1} \frac{4}{x+1} \neq 1$

d) $\lim_{x\to 3} \frac{3}{x-3} = 1$

Let $\varepsilon = 1$. Then for all values of δ, there will be values of x on the interval $(3-\delta, 3+\delta)$ such that $3 < x < 4$. For these values,

$$3 < x < 4$$
$$0 < x-3 < 1$$
$$1 < \tfrac{1}{x-3}$$
$$2 < \tfrac{3}{x-3} - 1$$

thus $|f(x) - 20| = \left|\tfrac{3}{x-3} - 1\right| > 2 > \varepsilon = 1$. Hence $\lim_{x\to 3} \frac{3}{x-3} \neq 1$

3. Determine whether the following statements are true or false, and prove your answer with the definition.

a) $\lim_{x\to 2}(3x^2 - x + 1) = 11$ True

Let $\varepsilon > 0$ be given.

$$\begin{aligned} |f(x) - L| &= |3x^2 - x + 1 - 11| \\ &= |3x+5||x-2| \\ &< 14\delta = \varepsilon \end{aligned}$$

Hyp: $0 < |x-2| < \delta$

Let $\delta \le 1$, then $-1 < x-2 < 1$

$$1 < x < 3$$
$$8 < 3x+5 < 14$$

If $\delta = \min\left\{1, \frac{\varepsilon}{14}\right\}$ then $0 < |x-2| < \delta \Rightarrow |f(x) - 11| < \varepsilon$, ergo $\lim_{x\to 2}(3x^2 - x + 1) = 11$

b) $\lim_{x\to -2} \frac{1}{x^2 - 4} = 1$ False

Let $\varepsilon = 1$. Then for all values of δ, there will be values of x on the interval $(-2-\delta, -2+\delta)$ such that $-2 < x < 0$. For these values,

$$-2 < x < 0$$
$$0 < x^2 < 4$$
$$-4 < x^2 - 4 < 0$$
$$\tfrac{1}{x^2-4} < \tfrac{-1}{4}$$
$$\tfrac{5}{4} < \tfrac{1}{x^2-4} - 1$$

thus $|f(x) - 1| = \left|\frac{1}{x^2-4} - 1\right| > \frac{5}{4} > \varepsilon = 1$. Hence $\lim_{x\to -2} \frac{1}{x^2-4} \neq 1$

c) $\lim_{x\to 2} f(x) = 1$ where $f(x) = \begin{cases} 2x-1 & x \neq 2 \\ 1 & x = 2 \end{cases}$ False

Let $\varepsilon = 1$. Then for all values of δ, there will be values of x on the interval $(2-\delta, 2+\delta)$ such that $x > 2$. For these values,

$$x > 2$$
$$2x - 2 > 2$$

thus $|f(x) - 1| = |2x - 1 - 1| > 2 > \varepsilon = 1$. Hence $\lim_{x\to 2} f(x) \neq 1$

d) $\lim_{x\to 2} f(x) = 3$ where $f(x) = \begin{cases} x^2 - 1 & x < 1 \\ x - 1 & x \geq 1 \end{cases}$ False

Let $\varepsilon = 1$. Then for all values of δ, there will be values of x on the interval $(2-\delta, 2+\delta)$ such that $2 < x < 3$. For these values,

$$2 < x < 3$$
$$-2 < x - 4 < -1$$
$$1 < |x - 4| < 2$$

thus $|f(x) - 3| = |x - 1 - 3| > \varepsilon = 1$. Hence $\lim_{x\to 2} f(x) \neq 3$

4. Show that the given limits do not exist.

a) $\lim_{x\to 0} f(x)$ where $f(x)=\begin{cases} 0 & x<0 \\ 1 & x\geq 0 \end{cases}$

Suppose $\lim_{x\to 0} f(x)=L$.

1. If $L\geq \frac{1}{2}$, then let $\varepsilon=\frac{1}{4}$. For all values of δ, there will be values of x on the interval $(-\delta,\delta)$ such that $x<0$. For these values,

$$\begin{aligned} |f(x)-L| &= |0-L| \\ &= L\geq \tfrac{1}{2}>\varepsilon=\tfrac{1}{4} \end{aligned}$$

2. If $L<\frac{1}{2}$, then let $\varepsilon=\frac{1}{4}$. For all values of δ, there will be values of x on the interval $(-\delta,\delta)$ such that $x>0$. For these values,

$$\begin{aligned} |f(x)-L| &= |1-L| \\ &> \tfrac{1}{2}>\varepsilon=\tfrac{1}{4} \end{aligned} \qquad \text{since} \quad \begin{aligned} L &< \tfrac{1}{2} \\ -L &> \tfrac{-1}{2} \\ 1-L &> \tfrac{1}{2} \end{aligned}$$

Thus, for any L, if $\varepsilon=\frac{1}{4}$, then $|f(x)-L|>\varepsilon$ for any δ.

b) $\lim_{x\to 0} f(x)$ where $f(x)=\begin{cases} x^2 & x\leq 0 \\ x-1 & x>0 \end{cases}$

Suppose $\lim_{x\to 0} f(x)=L$.

1. If $L\geq \frac{-1}{2}$, then let $\varepsilon=\frac{1}{4}$. For all values of δ, there will be values of x on the interval $(-\delta,\delta)$ such that $0<x<\frac{1}{4}$. For these values,

$$\begin{aligned} 0<x&<\tfrac{1}{4} \\ -1<x-1&<\tfrac{-3}{4} \\ x-1-L&<\tfrac{-3}{4}-L \\ x-1-L&<\tfrac{-3}{4}-\left(\tfrac{-1}{2}\right)=\tfrac{-1}{4} \qquad \text{since } \tfrac{-1}{2}\leq L \\ \tfrac{1}{4}&<|x-1-L| \end{aligned}$$

and thus $|f(x)-L|=|x-1-L|>\varepsilon=\frac{1}{4}$

2. If $L<\frac{-1}{2}$, then let $\varepsilon=\frac{1}{4}$. For all values of δ, there will be values of x on the interval $(-\delta,\delta)$ such that $x<0$. For these values,

$$\begin{aligned} x&<0 \\ x^2&>0 \\ x^2-L&>-L>\tfrac{1}{2} \end{aligned}$$

And thus $|f(x)-L|=|x^2-L|>\frac{1}{2}>\varepsilon=\frac{1}{4}$

Thus, for any L, if $\varepsilon=\frac{1}{4}$, then $|f(x)-L|>\varepsilon$ for any δ.

5. Use the definition of a limit to prove the following statements.

a) $\lim_{x\to 2^+}(2x+1)=5$

Let $\varepsilon>0$ be given.

$$\begin{aligned}|f(x)-L|&=|2x+1-5|\\&=2|x-2|\\&<2\delta=\varepsilon\end{aligned}$$

Hyp: $0<x-2<\delta$

If $\delta=\frac{\varepsilon}{2}$ then $0<x-2<\delta \Rightarrow |f(x)-5|<\varepsilon$, ergo $\lim_{x\to 2^+}(2x+1)=5$

b) $\lim_{x\to 1^-}(x^2+3x)=4$

Let $\varepsilon>0$ be given.

$$\begin{aligned}|f(x)-L|&=|x^2+3x-4|\\&=|x+4||x-1|\\&<5\delta=\varepsilon\end{aligned}$$

Hyp: $-\delta<x-1<0$

Let $\delta\le 1$, then $0<-(x-1)<1$

$$-5<-x-4<-4$$

$$4<x+4<5$$

If $\delta=\min\{1,\frac{\varepsilon}{5}\}$ then $-\delta<x-1<0 \Rightarrow |f(x)-4|<\varepsilon$, ergo $\lim_{x\to 1^-}(x^2+3x)=4$

c) $\lim_{x\to 1^-}f(x)=-1$ where $f(x)=\begin{cases}2x^2-3 & x<1\\ 5x+1 & x\ge 1\end{cases}$

Let $\varepsilon>0$ be given.

$$\begin{aligned}|f(x)-L|&=|2x^2-3+1|\\&=2|x+1||x-1|\\&<2\cdot 2\delta=\varepsilon\end{aligned}$$

Hyp: $-\delta<x-1<0$

$$1-\delta<x<1$$

Let $\delta\le 1$, then $0<-(x-1)<1$

$$-2<-x-1<-1$$

$$1<x+1<2$$

If $\delta=\min\{1,\frac{\varepsilon}{4}\}$ then $-\delta<x-1<0 \Rightarrow |f(x)-5|<\varepsilon$, ergo $\lim_{x\to 1^-}f(x)=5$

d) $\lim_{x\to 2^-}[\![x]\!]=1$

Let $\varepsilon>0$ be given.

$$\begin{aligned}|f(x)-L|&=|[\![x]\!]-1|\\&=|1-1|=0<\varepsilon\end{aligned}$$

Hyp: $-\delta<x-2<0$

$$2-\delta<x<2$$

Let $\delta\le 1$, then $1<x<2$

If $\delta=1$ then $-\delta<x-2<0 \Rightarrow |f(x)-1|<\varepsilon$, ergo $\lim_{x\to 2^-}[\![x]\!]=1$

e) $\lim_{x\to 2^+}\sqrt{x-2}=0$

Let $\varepsilon>0$ be given.

$$|f(x)-L|=|\sqrt{x-2}|<\sqrt{\delta}=\varepsilon$$

Hyp: $0<x-2<\delta$

If $\delta=\varepsilon^2$ then $0<x-2<\delta \Rightarrow |f(x)-0|<\varepsilon$, ergo $\lim_{x\to 2^+}\sqrt{x-2}=0$

f) $\lim\limits_{x\to 0^-} \frac{|x|}{x} = -1$

Let $\varepsilon > 0$ be given.

$$|f(x)-L| = \left|\tfrac{|x|}{x}+1\right| = \left|\tfrac{-x}{x}+1\right| = 0 < \varepsilon$$

Hyp: $-\delta < x < 0$

Thus $|x| = -x$

If $\delta = 1$ then $-\delta < x < 0 \Rightarrow |f(x)+1| < \varepsilon$, ergo $\lim\limits_{x\to 0^-} \frac{|x|}{x} = -1$

g) $\lim\limits_{x\to -\infty} \frac{4x}{2x+1} = 2$

Let $\varepsilon > 0$ be given.

$$|f(x)-L| = \left|\tfrac{4x}{2x+1}-2\right| = \left|\tfrac{-2}{2x+1}\right| < \tfrac{-2}{2M+1} = \varepsilon$$

Hyp: $x < M$

If $M < -\frac{1}{2}$

$$x < M < \tfrac{-1}{2}$$
$$2x+1 < 2M+1 < 0$$
$$0 < -(2M+1) < -(2x+1)$$
$$\tfrac{-1}{2x+1} < \tfrac{-1}{2M+1}$$

If $M = \frac{-1}{\varepsilon} - \frac{1}{2}$ then $x < M \Rightarrow |f(x)-2| < \varepsilon$, ergo $\lim\limits_{x\to -\infty} \frac{4x}{2x+1} = 2$

h) $\lim\limits_{x\to \infty} \frac{x+3}{x} = 1$

Let $\varepsilon > 0$ be given.

$$|f(x)-L| = \left|\tfrac{x+3}{x}-1\right| = \left|\tfrac{3}{x}\right| < \tfrac{3}{M} = \varepsilon$$

Hyp: $x > M$

If $M > 0$ then $\frac{1}{x} < \frac{1}{M}$

If $M = \frac{3}{\varepsilon}$ then $x > M \Rightarrow |f(x)-1| < \varepsilon$, ergo $\lim\limits_{x\to \infty} \frac{x+3}{x} = 1$

i) $\lim\limits_{x\to 5^+} \frac{4}{5-x} = -\infty$

Let $N < 0$ be given.

$$f(x) = \frac{4}{5-x} < \tfrac{-4}{\delta} = N$$

Hyp: $0 < x-5 < \delta$

$$\tfrac{1}{\delta} < \tfrac{1}{x-5}$$
$$\tfrac{4}{5-x} < \tfrac{-4}{\delta}$$

If $\delta = \frac{-4}{N}$ then $0 < x-5 < \delta \Rightarrow f(x) < N$, ergo $\lim\limits_{x\to 5^+} \frac{4}{5-x} = -\infty$

j) $\lim_{x\to\infty}\left(x^2+x\right)=\infty$

Let $N>0$ be given.

$f\left(x\right)=x^2+x>M^2+M=N$ Hyp: $x>M$

If $M>0$ then $x^2+x>M^2+M$

If $M=\frac{-1}{2}+\frac{1}{2}\sqrt{1+4N}$ then $x>M\Rightarrow f\left(x\right)>N$, ergo $\lim_{x\to\infty}\left(x^2+x\right)=\infty$

k) $\lim_{x\to-\infty}x^2=\infty$

Let $N>0$ be given.

$f\left(x\right)=x^2>M^2=N$ Hyp: $x<M$

If $M<0$ then $-x>-M$

$x^2>M^2$

If $M=-\sqrt{N}$ then $x<M\Rightarrow f\left(x\right)>N$, ergo $\lim_{x\to-\infty}x^2=\infty$

6. Use the definition of a limit to show that the following statements are false.

a) $\lim_{x\to2^+}\frac{1}{x-2}=2$

Let $\varepsilon=1$. Then for all values of δ, there will be values of x on the interval $\left(2,2+\delta\right)$ such that $2<x<\frac{9}{4}$. For these values,

$$2<x<\tfrac{9}{4}$$
$$0<x-2<\tfrac{1}{4}$$
$$4<\tfrac{1}{x-2}$$
$$2<\tfrac{1}{x-2}-2$$

thus $\left|f\left(x\right)-2\right|=\left|\frac{1}{x-2}-2\right|>2>\varepsilon=1$. Hence $\lim_{x\to2^+}\frac{1}{x-2}\neq2$

b) $\lim_{x\to1}\frac{x-1}{x^2-1}=\infty$

Let $N=10$. Then for all values of δ, there will be values of x on the interval $\left(1-\delta,1+\delta\right)$ such that $0<x<1$. For these values,

$$0<x<1$$
$$1<x+1<2$$
$$\tfrac{1}{2}<\tfrac{1}{x+1}<1$$

thus $f\left(x\right)=\frac{x-1}{x^2-1}=\frac{1}{x+1}<1<N=10$. Hence $\lim_{x\to1}\frac{x-1}{x^2-1}\neq\infty$

c) $\lim_{x\to 3^-} f(x) = -1$ where $f(x) = \begin{cases} x^2 & x < 3 \\ 2-x & x \geq 3 \end{cases}$

Let $\varepsilon = 1$. Then for all values of δ, there will be values of x on the interval $(3-\delta, 3)$ such that $2 < x < 3$. For these values,

$$2 < x < 3$$
$$4 < x^2 < 9$$
$$5 < x^2 + 1 < 10$$

thus $|f(x)+1| = |x^2+1| > 5 > \varepsilon = 1$. Hence $\lim_{x\to 3^-} f(x) \neq -1$

d) $\lim_{x\to 0^-} \frac{1}{x^3} = \infty$

Let $N = 10$. Then for all values of δ, there will be values of x on the interval $(-\delta, 0)$ such that $-1 < x < 0$. For these values,

$$-1 < x < 0$$
$$-1 < x^3 < 0$$
$$\tfrac{1}{x^3} < -1$$

thus $f(x) = \frac{1}{x^3} < -1 < N = 10$. Hence $\lim_{x\to 0^-} \frac{1}{x^3} \neq \infty$

e) $\lim_{x\to\infty} e^{-x} = \infty$

Let $N = 10$. Then for all values of M, there will be values of x on the interval (M, ∞) such that $x > 0$. For these values,

$$x > 0$$
$$-x < 0$$
$$e^{-x} < e^{-0} = 1$$

thus $f(x) = e^{-x} < 1 < N = 10$. Hence $\lim_{x\to\infty} e^{-x} \neq \infty$

f) $\lim_{x\to -\infty} \frac{x}{x+1} = -1$

Let $\varepsilon = 1$. Then for all values of M, there will be values of x on the interval $(-\infty, M)$ such that $x < -1$. For these values,

$$x < -1$$
$$x < x+1 < 0$$
$$1 < \tfrac{x}{x+1}$$
$$2 < \tfrac{x}{x+1} + 1$$

thus $|f(x)+1| = \left|\tfrac{x}{x+1} + 1\right| > 2 > \varepsilon = 1$. Hence $\lim_{x\to -\infty} \frac{x}{x+1} \neq -1$

g) $\lim_{x\to 4^+} \frac{3x}{x-4} = -\infty$

Let $N = -10$. Then for all values of δ, there will be values of x on the interval $(4, 4+\delta)$ such that $4 < x < 5$. For these values,

$$4 < x < 5$$
$$0 < x-4 < 1$$
$$\tfrac{1}{x-4} > 1$$
$$\tfrac{x}{x-4} > x > 4$$
$$\tfrac{3x}{x-4} > 12$$

thus $f(x) = \frac{3x}{x-4} > 12 > N = -10$. Hence $\lim_{x\to 4^+} \frac{3x}{x-4} \neq -\infty$

7. Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Prove the following properties of limits.

a) $\lim_{x\to a}\left[f(x) - g(x)\right] = L - M$

Since $\lim_{x\to a} f(x) = L$ then $\forall \varepsilon_1 > 0$, $\exists \delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon_1$$

Since $\lim_{x\to a} g(x) = M$ then $\forall \varepsilon_2 > 0$, $\exists \delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \Rightarrow |g(x) - M| < \varepsilon_2$$

Let $\varepsilon > 0$ be given.

$$\begin{aligned} |f(x) - g(x) - L + M| &= |f(x) - L - g(x) + M| \\ &\leq |f(x) - L| + |-g(x) + M| \\ &= |f(x) - L| + |g(x) - M| \\ &< \varepsilon_1 + \varepsilon_2 \end{aligned}$$

If we let $\varepsilon_1 = \frac{\varepsilon}{2}$ and $\varepsilon_2 = \frac{\varepsilon}{2}$, then and if $\delta = \min\{\delta_1, \delta_2\}$ then

$$0 < |x-a| < \delta \Rightarrow |f(x) + g(x) - L - M| < \varepsilon$$

b) $\lim_{x\to a} x^2 = a^2$

Let $\varepsilon > 0$ be given.

$$\begin{aligned} |f(x) - L| &= |x^2 - a^2| \\ &= |x-a||x+a| \\ &< (1 + 2|a|)\delta = \varepsilon \end{aligned}$$

Hyp: $0 < |x-a| < \delta$

Let $\delta < 1$, then

$$-1 < x - a < 1$$
$$-1 + 2a < x + a < 1 + 2a < 1 + 2|a|$$
$$|x+a| < 1 + 2|a|$$

If $\delta = \min\left\{1, \frac{\varepsilon}{1+2|a|}\right\}$ then $0 < |x-a| < \delta \Rightarrow |f(x) - a^2| < \varepsilon$, ergo $\lim_{x\to a} x^2 = a^2$

c) $\lim_{x\to a}\left[f(x)\right]^2 = L^2$ *Hint*: Use the boundness theorem

Since $\lim_{x\to a} f(x) = L$ then $\forall \varepsilon_1 > 0$, $\exists \delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon_1$$

Also, since $\lim_{x\to a} f(x)$ exists, then $\exists N > 0$ and $\exists \delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \Rightarrow |f(x)| < N$$

That is, such that $|f(x)+L| \le |f(x)| + |L| < N + |L|$

Let $\varepsilon > 0$ be given.

$$\left|f(x)^2 - L^2\right| = |f(x)-L||f(x)+L|$$

$$< \varepsilon_1\left(N + |L|\right)$$

If we let $\varepsilon_1 = \frac{\varepsilon}{N+|L|}$, then and if $\delta = \min\{\delta_1, \delta_2\}$ then

$$0 < |x-a| < \delta \Rightarrow \left|f(x)^2 - L^2\right| < \varepsilon$$