MATHEMATICS 201-BNK-05
Advanced Calculus
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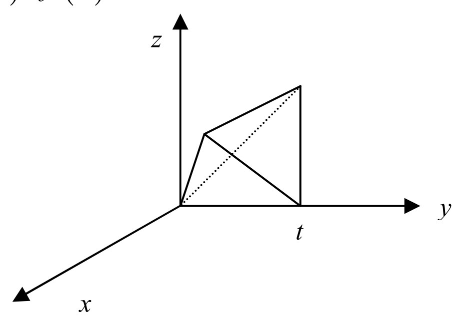
Winter 2011

Assignment #4
SOLUTIONS

This assignment is due ***Wednesday April 27, 2011*** at the beginning of the class.
Complete solutions with exact answers are expected, presented in a neat and legible manner.

Question 1 (8 points)

If f is continuous, show that $\int_0^t \int_0^y \int_0^z f(x)\,dxdzdy = \frac{1}{2}\int_0^t (x-t)^2 f(x)\,dx$.



$$\begin{aligned}
\int_0^t \int_0^y \int_0^z f(x)\,dxdzdy &= \int_0^t \int_x^t \int_z^t f(x)\,dydzdx \\
&= \int_0^t \int_x^t \left[f(x)\,y \right]_z^t dzdx \\
&= \int_0^t \int_x^t f(x)(t-z)\,dzdx \\
&= \int_0^t \left[f(x)\left(tz - \tfrac{1}{2}z^2\right) \right]_x^t dx \\
&= \int_0^t f(x)\left(t^2 - \tfrac{1}{2}t^2 - xt + \tfrac{1}{2}x^2\right) dx \\
&= \tfrac{1}{2}\int_0^t (x-t)^2 f(x)\,dx
\end{aligned}$$

Question 2 (8 points)

Find the center of mass for the solid that is outside the cylinder $x^2+y^2=3$ and inside the cylinder $x^2+y^2=2x$, bounded above and below by the planes $x-z=0$ and $x+z=0$, and with a density given by $\delta(x,y,z)=\dfrac{1}{\sqrt{x^2+y^2}}$.

$$\begin{aligned}
M &= \iiint_S \delta(x,y,z)\,dV = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\int_{\sqrt{3}}^{2\cos\theta}\int_{-r\cos\theta}^{r\cos\theta}\frac{1}{r}r\,dz\,dr\,d\theta \\
&= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\int_{\sqrt{3}}^{2\cos\theta}\int_{-r\cos\theta}^{r\cos\theta}dz\,dr\,d\theta \\
&= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\int_{\sqrt{3}}^{2\cos\theta}2r\cos\theta\,dr\,d\theta \\
&= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\left[r^2\cos\theta\right]_{\sqrt{3}}^{2\cos\theta}d\theta \\
&= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\left(4\cos^3\theta-3\cos\theta\right)d\theta \\
&= \int_{\frac{-\pi}{6}}^{\frac{\pi}{6}}\left(\cos\theta-4\cos\theta\sin^2\theta\right)d\theta \\
&= \left[\sin\theta-\tfrac{4}{3}\sin^3\theta\right]_{\frac{-\pi}{6}}^{\frac{\pi}{6}} \\
&= \frac{2}{3}
\end{aligned}$$

$$\begin{aligned}
M_{yz} &= \iiint_S x\delta(x,y,z)\,dV = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\int_{\sqrt{3}}^{2\cos\theta}\int_{-r\cos\theta}^{r\cos\theta}r\cos\theta\,dz\,dr\,d\theta \\
&= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\int_{\sqrt{3}}^{2\cos\theta}2r^2\cos^2\theta\,dr\,d\theta \\
&= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\left[\tfrac{2}{3}r^3\cos^2\theta\right]_{\sqrt{3}}^{2\cos\theta}d\theta \\
&= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\left(\tfrac{16}{3}\cos^5\theta-2\sqrt{3}\cos^2\theta\right)d\theta \\
&= \int_{\frac{-\pi}{6}}^{\frac{\pi}{6}}\left(\tfrac{16}{3}\cos\theta\left(1-\sin^2\theta\right)^2-\sqrt{3}-\sqrt{3}\cos 2\theta\right)d\theta \\
&= \left[\tfrac{16}{3}\left(\sin\theta-\tfrac{2}{3}\sin^3\theta+\tfrac{1}{5}\sin^5\theta\right)-\sqrt{3}\theta-\tfrac{\sqrt{3}}{2}\sin 2\theta\right]_{\frac{-\pi}{6}}^{\frac{\pi}{6}} \\
&= \frac{271}{90}-\frac{\sqrt{3}}{3}\pi
\end{aligned}$$

By symmetry $\bar{y}=\bar{z}=0$

Ergo, the center of mass is: $(\bar{x},\bar{y},\bar{z})=\left(\tfrac{271}{60}-\tfrac{\sqrt{3}}{2}\pi,0,0\right)$

Question 3 (8 points)

Consider the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

a) Find the volume of the ellipsoid using the following change of variables.

$$x = ar\sin\phi\cos\theta$$
$$y = br\sin\phi\sin\theta$$
$$z = cr\cos\phi$$

$$J(r,\phi,\theta)=\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta}\end{vmatrix}$$

$$=\begin{vmatrix}a\sin\phi\cos\theta & ar\cos\phi\cos\theta & -ar\sin\phi\sin\theta\\ b\sin\phi\sin\theta & br\cos\phi\sin\theta & br\sin\phi\cos\theta\\ c\cos\phi & -cr\sin\phi & 0\end{vmatrix}$$

$$=c\cos\phi\left(abr^2\sin\phi\cos\phi\cos^2\theta+abr^2\sin\phi\cos\phi\sin^2\theta\right)$$
$$+cr\sin\phi\left(abr\sin^2\phi\cos^2\theta+abr\sin^2\phi\sin^2\theta\right)$$
$$=abcr^2\sin\phi\cos^2\phi+abcr^2\sin\phi\sin^2\phi$$
$$=abcr^2\sin\phi$$

$$V=\iiint_S dV=\int_0^{2\pi}\int_0^{\pi}\int_0^1 abcr^2\sin\phi\,dr\,d\phi\,d\theta$$
$$=\tfrac{1}{3}abc\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta$$
$$=\tfrac{2}{3}abc\int_0^{2\pi}d\theta$$
$$=\frac{4\pi abc}{3}$$

b) Earth is not a perfect sphere; rotation has resulted in fattening at the poles So the shape can be approximated by an ellipsoid with $a=b=6378$ km and $c=6356$ km. Use the answer in (b) to estimate the volume of the earth.

$$V_{\text{earth}}=\tfrac{4}{3}(6378)^2(6356)\pi\simeq 1.083\times10^{12}\ \text{km}^3$$

Question 4 (8 points)
Solve the following differential equations.

$$x\frac{dy}{dx}+y=x\sqrt{y}\arctan\sqrt{x}$$

$\frac{dy}{dx}+\frac{1}{x}y=\sqrt{y}\arctan\sqrt{x}$ Bernouilli differential equation

Let $z=\sqrt{y}$

$$\frac{dz}{dx}=\frac{1}{2\sqrt{y}}\frac{dy}{dx}$$

Then $\frac{1}{2\sqrt{y}}\frac{dy}{dx}+\frac{1}{2x}\sqrt{y}=\frac{1}{2}\arctan\sqrt{x}$

$\frac{dz}{dx}+\frac{1}{2x}z=\frac{1}{2}\arctan\sqrt{x}$ Linear D.E.

$$\mu=e^{\int\frac{1}{2x}dx}=e^{\frac{1}{2}\ln x}=e^{\ln\sqrt{x}}=\sqrt{x}$$

Hence the solution is

$$\sqrt{x}z=\int\tfrac{1}{2}\sqrt{x}\arctan\sqrt{x}\,dx$$

$u=\arctan\sqrt{x}$ $v=\frac{1}{3}x^{\frac{3}{2}}$

$du=\frac{1}{2\sqrt{x}(1+x)}dx$ $dv=\frac{1}{2}\sqrt{x}dx$

$$\sqrt{x}\sqrt{y}=\tfrac{1}{3}x^{\frac{3}{2}}\arctan\sqrt{x}-\tfrac{1}{6}\int\frac{x}{1+x}dx$$

$$\sqrt{x}\sqrt{y}=\tfrac{1}{3}x^{\frac{3}{2}}\arctan\sqrt{x}-\tfrac{1}{6}\int\left(1-\frac{1}{1+x}\right)dx$$

$$\sqrt{x}\sqrt{y}=\tfrac{1}{3}x^{\frac{3}{2}}\arctan\sqrt{x}-\tfrac{1}{6}x+\tfrac{1}{6}\ln|1+x|+C$$

$$\sqrt{y}=\tfrac{1}{3}x\arctan\sqrt{x}-\tfrac{1}{6}\sqrt{x}+\frac{\ln|1+x|}{6\sqrt{x}}+\frac{C}{\sqrt{x}}$$

$$y=\left(\tfrac{1}{3}x\arctan\sqrt{x}-\tfrac{1}{6}\sqrt{x}+\frac{\ln|1+x|}{6\sqrt{x}}+\frac{C}{\sqrt{x}}\right)^2$$

Question 5 (9 points)

Consider a differential equation of the form $yf\left(xy\right)dx+xg\left(xy\right)dy=0$ where $f\left(xy\right)\neq g\left(xy\right)$.

a) Show that multiplying the above equation by the following integrating factor yields an exact differential equation.

$$\frac{1}{xy\left(f\left(xy\right)-g\left(xy\right)\right)}$$

$$\frac{f\left(xy\right)}{x\left(f\left(xy\right)-g\left(xy\right)\right)}dx+\frac{g\left(xy\right)}{y\left(f\left(xy\right)-g\left(xy\right)\right)}dy=0$$

$$\frac{\partial M}{\partial y}=\frac{xf'\left(xy\right)x\left(f\left(xy\right)-g\left(xy\right)\right)-f\left(xy\right)\left(x^2f'\left(xy\right)-x^2g'\left(xy\right)\right)}{x^2\left(f\left(xy\right)-g\left(xy\right)\right)^2}$$

$$=\frac{-f'\left(xy\right)g\left(xy\right)+f\left(xy\right)g'\left(xy\right)}{\left(f\left(xy\right)-g\left(xy\right)\right)^2}$$

$$\frac{\partial N}{\partial x}=\frac{yg'\left(xy\right)y\left(g\left(xy\right)-f\left(xy\right)\right)-g\left(xy\right)\left(y^2f'\left(xy\right)-y^2g'\left(xy\right)\right)}{y^2\left(f\left(xy\right)-g\left(xy\right)\right)^2}$$

$$=\frac{g'\left(xy\right)f\left(xy\right)-f'\left(xy\right)g\left(xy\right)}{\left(f\left(xy\right)-g\left(xy\right)\right)^2}$$

Hence $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ and the differential equation is exact.

b) Use (a) to solve the following differential equation.

$$y\left(2xy+1\right)dx+x\left(1+2xy-x^3y^3\right)dy=0 \qquad y\left(2\right)=1$$

$$f\left(xy\right)=2xy+1$$

$$g\left(xy\right)=1+2xy-x^3y^3$$

Hence the integrating factor is $\frac{1}{xy\left(2xy+1-1-2xy+x^3y^3\right)}=\frac{1}{x^4y^4}$

Giving the exact differential equation

$$\left(\frac{2}{x^3y^2}+\frac{1}{x^4y^3}\right)dx+\left(\frac{1}{x^3y^4}+\frac{2}{x^2y^3}-\frac{1}{y}\right)dy=0$$

$$f\left(x,y\right)=\int\left(\frac{2}{x^3y^2}+\frac{1}{x^4y^3}\right)dx+\phi\left(y\right)$$

$$=\frac{-1}{x^2y^2}-\frac{1}{3x^3y^3}+\phi\left(y\right)$$

$$\frac{\partial f}{\partial y}=\frac{2}{x^2y^3}+\frac{1}{x^3y^4}+\phi'\left(y\right)=\frac{1}{x^3y^4}+\frac{2}{x^2y^3}-\frac{1}{y}$$

$$\phi'(y) = -\frac{1}{y}$$

$$\phi(y) = -\ln y + C$$

Hence $\frac{-1}{x^2y^2} - \frac{1}{3x^3y^3} - \ln y = C$

$$3xy + 1 + 3x^3y^3 \ln y = 3x^3y^3K$$

$$y(2) = 1$$

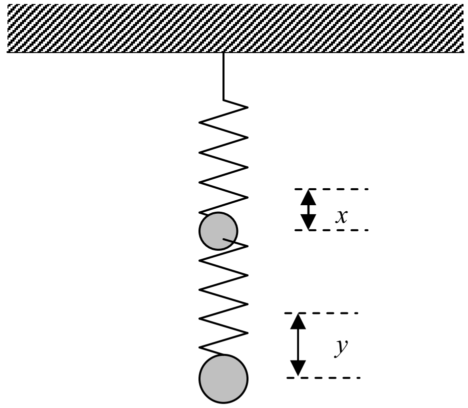
$$6 + 1 + 24\ln 1 = 24K$$

$$K = \tfrac{7}{24}$$

Ergo, the solution is $24xy + 8 + 24x^3y^3 \ln y = 7x^3y^3$

Question 6 (9 points)

Suppose that two objects of mass m_1 and m_2 are is attached to elastic springs suspended from the ceiling.



If x and y denote the displacement of the masses m_1 and m_2 from their equilibrium positions, g to Hooke's law and Newton's Second law of motion gives the following system of differential equations

$$m_1 \frac{d^2x}{dt^2} = k_2(y-x) - k_1 x$$

$$m_2 \frac{d^2y}{dt^2} = -k_2(y-x)$$

where friction is negligible. Suppose that $m_1 = 2$ kg, $m_2 = 1$ kg, $k_1 = 4$ kg/m and $k_2 = 2$ kg/m. Also, suppose that $x(0) = 2$, $x'(0) = 0$, $y(0) = 3$ and $y'(0) = 0$.

a) Combine the two differential equations together to obtain the following differential equation in terms of x only. *Hint*: *Differentiate the first equation twice to get a new differential equation of degree 4. Then substitute the y's using the two equations given above.*

$$\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 4x = 0$$

$$2\frac{d^2x}{dt^2} = 2(y-x) - 4x \qquad \frac{d^2x}{dt^2} = -3x + y \qquad y = \frac{d^2x}{dt^2} + 3x$$

$$\frac{d^2y}{dt^2} = -2(y-x) \qquad \frac{d^2y}{dt^2} = 2x - 2y$$

$$\frac{d^3x}{dt^3} = -3\frac{dx}{dt} + \frac{dy}{dt}$$
$$\frac{d^4x}{dt^4} = -3\frac{d^2x}{dt^2} + \frac{d^2y}{dt^2}$$
$$\frac{d^4x}{dt^4} = -3\frac{d^2x}{dt^2} + 2x - 2y$$
$$\frac{d^4x}{dt^4} = -3\frac{d^2x}{dt^2} + 2x - 2\left(\frac{d^2x}{dt^2} + 3x\right)$$
$$\frac{d^4x}{dt^4} = -5\frac{d^2x}{dt^2} - 4x$$
$$\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 4x = 0$$

b) Solve the above differential equation to a general solution for $x(t)$.

Characteristic equation: $r^4 + 5r^2 + 4 = 0$

$$(r^2+1)(r^2+4) = 0$$
$$r = \pm i, \pm 2i$$
$$x(t) = c_1 \cos t + c_2 \sin t + c_3 \cos(2t) + c_4 \sin(2t)$$

c) Use the answer found in (b) to find a general solution for $y(t)$. *Hint: Replace the solution $x(t)$ in the first equation given above.*

$$\frac{dx}{dt} = -c_1 \sin t + c_2 \cos t - 2c_3 \sin(2t) + 2c_4 \cos(2t)$$
$$y = \frac{d^2x}{dt^2} + 3x$$
$$y(t) = -c_1 \cos t - c_2 \sin t - 4c_3 \cos(2t) - 4c_4 \sin(2t)$$
$$+3\left(c_1 \cos t + c_2 \sin t + c_3 \cos(2t) + c_4 \sin(2t)\right)$$
$$= 2c_1 \cos t + 2c_2 \sin t - c_3 \cos(2t) - c_4 \sin(2t)$$

d) Find the particular solutions for $x(t)$ and $y(t)$ using the given initial conditions.

$$x(0) = c_1 + c_3 = 2$$
$$y(0) = \underline{2c_1 - c_3 = 3}$$
$$3c_1 = 5$$
$$c_1 = \tfrac{5}{3},\ c_3 = \tfrac{1}{3}$$
$$x'(t) = -c_1 \sin t + c_2 \cos t - 2c_3 \sin(2t) + 2c_4 \cos(2t)$$
$$y'(t) = -2c_1 \sin t + c_2 \cos t + 2c_3 \sin(2t) - 2c_4 \cos(2t)$$

$$x'(0) = c_2 + 2c_4 = 0$$
$$y'(0) = \underline{c_2 - 2c_4 = 0}$$
$$4c_4 = 0$$
$$c_4 = 0,\ \ c_2 = 0$$

Hence $$x(t) = \tfrac{5}{3}\cos t + \tfrac{1}{3}\cos(2t)$$
$$y(t) = \tfrac{10}{3}\cos t - \tfrac{1}{3}\cos(2t)$$