



MATHEMATICS 201-BNK-05

Advanced Calculus

Martin Huard

Winter 2011

Assignment #2 SOLUTIONS

This assignment is due **Tuesday March 1, 2011** at the beginning of the class.
Complete solutions with exact answers are expected, presented in a neat and legible manner.

For questions involving Maple, a print-out of your work is expected, where your name is written in the Worksheet, each question is clearly labeled, and the answers are clearly presented. Also, you must copy your file in my "TEST" subfolder (W:\Tests\mhuard\Advanced Calculus \Assignment 2), where your name should be included in the name of the file (for example: Assignment 2 – Your Name).

Question 1 (8 points)

Calculate the following limit, and use the definition with ε and δ to prove your answer.

$$\lim_{(x,y) \rightarrow (-2,1)} (x^2 + xy^2 - 4)$$

Let $\varepsilon > 0$ be given.

$$|f(x,y) - (-2)| = |x^2 + xy^2 - 2|$$

$$= |x^2 + xy^2 - x + x - 2|$$

$$= |x^2 + x - 2 + x(y^2 - 1)|$$

$$= |(x+2)(x-1) + x(y-1)(y+1)|$$

$$\leq |x+2||x-1| + x|y-1||y+1|$$

$$< 4\delta + 3(3)\delta = 13\delta = \varepsilon$$

$$\text{Hyp: } 0 < \sqrt{(x+2)^2 + (y-1)^2} < \delta$$

This implies that

$$|x+2| < \delta \text{ and } |y-1| < \delta$$

Let $\delta \leq 1$

$$|x+2| < 1$$

$$|y-1| < 1$$

$$-1 < x+2 < 1$$

$$-1 < y-1 < 1$$

$$-3 < x < -1$$

$$2 < y+1 < 3$$

$$-4 < x-1 < -2$$

$$2 < -(x-1) < 4$$

If $\delta = \min\{1, \frac{\varepsilon}{13}\}$, then $0 < \sqrt{(x+2)^2 + (y-1)^2} < \delta \Rightarrow |f(x,y) + 2| < \varepsilon$

$$\text{thus } \lim_{(x,y) \rightarrow (-2,1)} (x^2 + xy^2 - 4) = -2$$

Question 2 (8 points)

Prove that the function $f(x, y) = 4x^2 - 5xy + 6y^3$ is differentiable at $(2, 3)$ using the definition.

$$f_x(x, y) = 8x - 5y \quad f_x(2, 3) = 1$$

$$f_y(x, y) = -5x + 18y^2 \quad f_y(2, 3) = 152$$

$$\text{Hence we have: } f_x(2, 3)\Delta x + f_y(2, 3)\Delta y = \Delta x + 152\Delta y$$

$$\Delta f = f(2 + \Delta x, 3 + \Delta y) - f(2, 3)$$

$$= 4(2 + \Delta x)^2 - 5(2 + \Delta x)(3 + \Delta y) + 6(3 + \Delta y)^3 - 148$$

$$= 16 + 16\Delta x + 4\Delta x^2 - 30 - 15\Delta x - 10\Delta y - 5\Delta x\Delta y + 162 + 162\Delta y + 54\Delta y^2 + 6\Delta y^3 - 148$$

$$= \Delta x + 152\Delta y + 4\Delta x^2 - 5\Delta x\Delta y + 54\Delta y^2 + 6\Delta y^3$$

$$= f_x(2, 3)\Delta x + f_y(2, 3)\Delta y + (4\Delta x - 5\Delta y)\Delta x + (54\Delta y + 6\Delta y^2)\Delta y$$

$$= f_x(2, 3)\Delta x + f_y(2, 3)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$$

Thus if $\varepsilon_1 = 4\Delta x - 5\Delta y$ and $\varepsilon_2 = 54\Delta y + 6\Delta y^2$, then

$$\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \varepsilon_1 = \lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} (4\Delta x - 5\Delta y) = 0$$

$$\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \varepsilon_2 = \lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} (54\Delta y + 6\Delta y^2) = 0$$

Ergo, f is differentiable at $(2, 3)$.

Question 3 (10 points)

If f has continuous partial derivatives, show that the function $w = f(e^r \cos \theta, e^r \sin \theta)$ satisfies

$$\frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 w}{\partial \theta^2} = (x^2 + y^2) \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)$$

Let $x = e^r \cos \theta$, then $w = f(x, y)$

$$y = e^r \sin \theta$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} e^r \cos \theta + \frac{\partial w}{\partial y} e^r \sin \theta$$

$$\begin{aligned} \frac{\partial^2 w}{\partial r^2} &= \left(\frac{\partial^2 w}{\partial x^2} e^r \cos \theta + \frac{\partial^2 w}{\partial y \partial x} e^r \sin \theta \right) e^r \cos \theta + \frac{\partial w}{\partial x} e^r \cos \theta \\ &\quad + \left(\frac{\partial^2 w}{\partial x \partial y} e^r \cos \theta + \frac{\partial^2 w}{\partial y^2} e^r \sin \theta \right) e^r \sin \theta + \frac{\partial w}{\partial y} e^r \sin \theta \\ &= \frac{\partial^2 w}{\partial x^2} e^{2r} \cos^2 \theta + 2 \frac{\partial^2 w}{\partial x \partial y} e^{2r} \sin \theta \cos \theta + \frac{\partial^2 w}{\partial y^2} e^{2r} \sin^2 \theta + \frac{\partial w}{\partial x} e^r \cos \theta + \frac{\partial w}{\partial y} e^r \sin \theta \end{aligned}$$

$$\frac{\partial w}{\partial \theta} = -\frac{\partial w}{\partial x} e^r \sin \theta + \frac{\partial w}{\partial y} e^r \cos \theta$$

$$\begin{aligned} \frac{\partial^2 w}{\partial \theta^2} &= - \left(-\frac{\partial^2 w}{\partial x^2} e^r \sin \theta + \frac{\partial^2 w}{\partial x \partial y} e^r \cos \theta \right) e^r \sin \theta - \frac{\partial w}{\partial x} e^r \cos \theta \\ &\quad + \left(-\frac{\partial^2 w}{\partial y \partial x} e^r \sin \theta + \frac{\partial^2 w}{\partial y^2} e^r \cos \theta \right) e^r \cos \theta - \frac{\partial w}{\partial y} e^r \sin \theta \\ &= \frac{\partial^2 w}{\partial x^2} e^{2r} \sin^2 \theta - 2 \frac{\partial^2 w}{\partial x \partial y} e^{2r} \sin \theta \cos \theta + \frac{\partial^2 w}{\partial y^2} e^{2r} \cos^2 \theta - \frac{\partial w}{\partial x} e^r \cos \theta - \frac{\partial w}{\partial y} e^r \sin \theta \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 w}{\partial r^2} + \frac{\partial^2 w}{\partial \theta^2} &= \frac{\partial^2 w}{\partial x^2} e^{2r} (\sin^2 \theta + \cos^2 \theta) + \frac{\partial^2 w}{\partial y^2} e^{2r} (\sin^2 \theta + \cos^2 \theta) \\ &= e^{2r} (\sin^2 \theta + \cos^2 \theta) \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \\ &= (e^{2r} \sin^2 \theta + e^{2r} \cos^2 \theta) \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \\ &= (x^2 + y^2) \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \end{aligned}$$

Question 4 (14 points)

Consider $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$ and $z = \rho \cos \phi$.

a) Find $\frac{\partial \rho}{\partial x}$, $\frac{\partial \rho}{\partial y}$ and $\frac{\partial \phi}{\partial z}$.

$$\text{Let } F(x, y, z, \rho, \theta, \phi) = \rho \sin \phi \cos \theta - x$$

$$G(x, y, z, \rho, \theta, \phi) = \rho \sin \phi \sin \theta - y$$

$$H(x, y, z, \rho, \theta, \phi) = \rho \cos \phi - z$$

$$\begin{aligned} \frac{\partial(F, G, H)}{\partial(\rho, \theta, \phi)} &= \begin{vmatrix} \sin \phi \cos \theta & -\rho \sin \phi \sin \theta & \rho \cos \phi \cos \theta \\ \sin \phi \sin \theta & \rho \sin \phi \cos \theta & \rho \cos \phi \sin \theta \\ \cos \phi & 0 & -\rho \sin \phi \end{vmatrix} \\ &= \cos \phi \left(-\rho^2 \sin \phi \cos \phi \sin^2 \theta - \rho^2 \sin \phi \cos \phi \cos^2 \theta \right) \\ &\quad - \rho \sin \phi \left(\rho \sin^2 \phi \cos^2 \theta + \rho \sin^2 \phi \sin^2 \theta \right) \\ &= -\rho^2 \sin \phi \cos^2 \phi - \rho^2 \sin^3 \phi \\ &= -\rho^2 \sin \phi (\cos^2 \phi + \sin^2 \phi) \\ &= -\rho^2 \sin \phi \end{aligned}$$

$$\begin{aligned} \frac{\partial(F, G, H)}{\partial(x, \theta, \phi)} &= \begin{vmatrix} -1 & -\rho \sin \phi \sin \theta & \rho \cos \phi \cos \theta \\ 0 & \rho \sin \phi \cos \theta & \rho \cos \phi \sin \theta \\ 0 & 0 & -\rho \sin \phi \end{vmatrix} \\ &= -(-\rho^2 \sin^2 \phi \cos \theta) \\ &= \rho^2 \sin^2 \phi \cos \theta \end{aligned}$$

$$\frac{\partial \rho}{\partial x} = \frac{\frac{\partial(F, G, H)}{\partial(x, \theta, \phi)}}{\frac{\partial(F, G, H)}{\partial(\rho, \theta, \phi)}} = \frac{\rho^2 \sin^2 \phi \cos \theta}{-\rho^2 \sin \phi} = \sin \phi \cos \theta$$

$$\begin{aligned} \frac{\partial(F, G, H)}{\partial(y, \theta, \phi)} &= \begin{vmatrix} 0 & -\rho \sin \phi \sin \theta & \rho \cos \phi \cos \theta \\ -1 & \rho \sin \phi \cos \theta & \rho \cos \phi \sin \theta \\ 0 & 0 & -\rho \sin \phi \end{vmatrix} \\ &= -(-1)(\rho^2 \sin^2 \phi \sin \theta) \\ &= \rho^2 \sin^2 \phi \sin \theta \end{aligned}$$

$$\frac{\partial \rho}{\partial y} = \frac{\frac{\partial(F, G, H)}{\partial(y, \theta, \phi)}}{\frac{\partial(F, G, H)}{\partial(\rho, \theta, \phi)}} = \frac{\rho^2 \sin^2 \phi \sin \theta}{-\rho^2 \sin \phi} = \sin \phi \sin \theta$$

$$\begin{aligned} \frac{\partial(F, G, H)}{\partial(\rho, \theta, z)} &= \begin{vmatrix} \sin \phi \cos \theta & -\rho \sin \phi \sin \theta & 0 \\ \sin \phi \sin \theta & \rho \sin \phi \cos \theta & 0 \\ \cos \phi & 0 & -1 \end{vmatrix} \\ &= -(\rho \sin^2 \phi \cos^2 \theta + \rho \sin^2 \phi \sin^2 \theta) \\ &= -\rho \sin^2 \phi \\ \frac{\partial \rho}{\partial x} &= \frac{-\frac{\partial(F, G, H)}{\partial(x, \theta, \phi)}}{\frac{\partial(F, G, H)}{\partial(\rho, \theta, \phi)}} = \frac{-(-\rho \sin^2 \phi)}{-\rho^2 \sin \phi} = \frac{-\sin \phi}{\rho} \end{aligned}$$

b) Use your result from (a) to verify that $\left(\frac{\partial \rho}{\partial x}\right)^2 + \left(\frac{\partial \rho}{\partial y}\right)^2 = \rho^2 \left(\frac{\partial \phi}{\partial z}\right)^2$.

$$\begin{aligned} \left(\frac{\partial \rho}{\partial x}\right)^2 + \left(\frac{\partial \rho}{\partial y}\right)^2 &= \sin^2 \phi \cos^2 \theta + \sin^2 \phi \sin^2 \theta \\ &= \sin^2 \phi \\ &= \rho^2 \frac{\sin^2 \phi}{\rho^2} \\ &= \rho^2 \left(\frac{\partial \phi}{\partial z}\right)^2 \end{aligned}$$

c) Verify your answers from (a) and (b) with Maple.

MATHEMATICS 201-BNK-05
Advanced Calculus
Martin Huard
Winter 2011

Assignment 2 Solutions

Question 4
restart;
with(VectorCalculus):
f := r * sin(phi) * cos(theta) - x;
g := r * sin(phi) * sin(theta) - y;
h := r * cos(phi) - z;

$$\begin{aligned} & r \sin(\phi) \cos(\theta) - x \\ & r \sin(\phi) \sin(\theta) - y \\ & r \cos(\phi) - z \end{aligned} \tag{1}$$

$\frac{\partial(f, g, h)}{\partial(r, \theta, \phi)}$
M1, d1 := Jacobian([f, g, h], [r, theta, phi], 'determinant');

$$\begin{vmatrix} \sin(\phi) \cos(\theta) & -r \sin(\phi) \sin(\theta) & r \cos(\phi) \cos(\theta) \\ \sin(\phi) \sin(\theta) & r \sin(\phi) \cos(\theta) & r \cos(\phi) \sin(\theta) \\ \cos(\phi) & 0 & -r \sin(\phi) \end{vmatrix} = -\sin(\phi)^3 \cos(\theta)^2 r^2 - \sin(\phi)^3 \sin(\theta)^2 r^2 - r^2 \sin(\phi) \sin(\theta)^2 \cos(\phi)^2 - r^2 \sin(\phi) \cos(\theta)^2 \cos(\phi)^2$$

$$- \sin(\phi) r^2 \tag{2}$$

simplify(d1);

$\frac{\partial(f, g, h)}{\partial(x, \theta, \phi)}$
M2, d2 := Jacobian([f, g, h], [x, theta, phi], 'determinant');

$$\begin{vmatrix} -1 & -r \sin(\phi) \sin(\theta) & r \cos(\phi) \cos(\theta) \\ 0 & r \sin(\phi) \cos(\theta) & r \cos(\phi) \sin(\theta) \\ 0 & 0 & -r \sin(\phi) \end{vmatrix} = r^2 \sin(\phi)^2 \cos(\theta)$$

$$r^2 \sin(\phi)^2 \cos(\theta) \tag{4}$$

$\frac{\partial(f, g, h)}{\partial(r, \theta, \phi)}$
M3, d3 := Jacobian([f, g, h], [y, theta, phi], 'determinant');

$$\begin{vmatrix} 0 & -r \sin(\phi) \sin(\theta) & r \cos(\phi) \cos(\theta) \\ -1 & r \sin(\phi) \cos(\theta) & r \cos(\phi) \sin(\theta) \\ 0 & 0 & -r \sin(\phi) \end{vmatrix} = r^2 \sin(\phi)^2 \sin(\theta)$$

$$r^2 \sin(\phi)^2 \sin(\theta) \tag{5}$$

$$\begin{aligned}
 & \frac{\partial(\mathbf{f}, \mathbf{g}, \mathbf{h})}{\partial(r, \theta, z)} \\
 & M4, d4 := \text{Jacobian}([\mathbf{f}, \mathbf{g}, \mathbf{h}], [r, \theta, z], \text{'determinant'}); \\
 & \begin{vmatrix} \sin(\phi) \cos(\theta) & -r \sin(\phi) \sin(\theta) & 0 \\ \sin(\phi) \sin(\theta) & r \sin(\phi) \cos(\theta) & 0 \\ \cos(\phi) & 0 & -1 \end{vmatrix} \cdot -\sin(\phi)^2 \cos(\theta)^2 r - \sin(\phi)^2 \sin(\theta)^2 r \quad (6) \\
 & \text{simplify}(d4); \\
 & \qquad \qquad \qquad (-1 + \cos(\phi)^2) r \quad (7) \\
 & \frac{\partial r}{\partial x} \\
 & \text{simplify}\left(\frac{-d2}{d1}\right); \\
 & \qquad \qquad \qquad \sin(\phi) \cos(\theta) \quad (8) \\
 & \frac{\partial r}{\partial y} \\
 & \text{simplify}\left(\frac{-d3}{d1}\right); \\
 & \qquad \qquad \qquad \sin(\phi) \sin(\theta) \quad (9) \\
 & \frac{\partial \phi}{\partial z} \\
 & \text{simplify}\left(\frac{-d4}{d1}\right); \\
 & \qquad \qquad \qquad -\frac{\sin(\phi)}{r} \quad (10) \\
 & \text{(b) Verifying the identity} \\
 & LS = \text{simplify}\left(\left(\frac{-d2}{d1}\right)^2 + \left(\frac{-d3}{d1}\right)^2\right); \\
 & RS = \text{simplify}\left(r^2 \cdot \left(\frac{-d4}{d1}\right)^2\right) \\
 & \qquad \qquad \qquad LS = \sin(\phi)^2 \\
 & \qquad \qquad \qquad RS = \sin(\phi)^2 \quad (11)
 \end{aligned}$$

Question 5 (10 points)

Consider the surface $xyz = c$.

- a) Show that the product of the x -, y -, and z -intercepts of any tangent plane to the surface is a constant.

$$f(x, y, z) = xyz - c$$

$$\nabla f(x, y, z) = (yz, xz, xy)$$

$$\text{At } (x_0, y_0, z_0), \nabla f(x_0, y_0, z_0) = (y_0z_0, x_0z_0, x_0y_0)$$

the equation of the tangent plane is:

$$xy_0z_0 + x_0yz_0 + x_0y_0z = x_0y_0z_0 + x_0y_0z_0 + x_0y_0z_0$$

$$xy_0z_0 + x_0yz_0 + x_0y_0z = 3c$$

Hence, the x -, y -, and z -intercepts are $\frac{3c}{y_0z_0}$, $\frac{3c}{x_0z_0}$ and $\frac{3c}{x_0y_0}$ respectively,

$$\begin{aligned} \text{Thus, product of the intercepts} &= \frac{3c}{y_0z_0} \frac{3c}{x_0z_0} \frac{3c}{x_0y_0} \\ &= \frac{27c^3}{(x_0y_0z_0)^2} \\ &= \frac{27c^3}{c^2} = 27c \end{aligned}$$

- b) Assuming that $c = 6$, find the equation of the tangent plane and the normal line at the point $(3, 1, 2)$.

$$\nabla f(3, 1, 2) = (2, 6, 3)$$

$$\text{Equation of tangent plane: } 2x + 6y + 3z = 6 + 6 + 6$$

$$2x + 6y + 3z = 18$$

$$\text{Normal line: } (x, y, z) = (3, 1, 2) + t(2, 6, 3)$$

- c) Show that the two surfaces $xyz = 6$ and $(x-7)^2 + (y-13)^2 + (z-8)^2 = 196$ are tangent at the point $(3, 1, 2)$.

$$g(x, y, z) = (x-7)^2 + (y-13)^2 + (z-8)^2 - 196$$

$$\nabla g(x, y, z) = (2(x-7), 2(y-13), 2(z-8))$$

$$\nabla g(3, 1, 2) = (-8, -24, -12) = -4(2, 6, 3) = -4\nabla f(3, 1, 2)$$

$$\text{Equation of tangent plane: } 2x + 6y + 3z = 18$$

Hence the two surfaces have the same tangent plane, thus are parallel.

- d) Using Maple, sketch a graph of the surfaces $xyz = 6$ and

$(x-7)^2 + (y-13)^2 + (z-8)^2 = 196$, as well as the tangent plane and the normal line found in (b), all on the same screen.

Question 5

(d)

with (plots) :

```
surface1 := implicitplot3d(x·y·z = 6, x=-8..25, y=-8..25, z=-8..25, axes = normal, color = blue, numpoints = 1000) :
```

```
surface2 := implicitplot3d((x - 7)2 + (y - 13)2 + (z - 8)2 = 196, x=-8..25, y=-8..25, z=-8..25, axes = normal, color = red) :
```

```
tangentplane := implicitplot3d(2·x + 6·y + 3·z = 18, x=-8..25, y=-8..25, z=-8..25, axes = normal, color = khaki) :
```

```
normalline := spacecurve([3 + 2·t, 1 + 6·t, 2 + 3·t], t=-1..3, color = black, thickness = 2) :
```

```
display(surface1, surface2, tangentplane, normalline);
```

